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Conjugate Line Congruences of the Third Order Defined by a Family of Quadrics.

BY HELEN BREWSTER OWENS.

INTRODUCTION.

The first systematic study of algebraic congruences was made by Kummer, who derived most of the forms of those of order 2. His work is both synthetic and analytic, and is a model of elegance and directness. All of these congruences can be arranged on a system of quadric surfaces in such a way that only one system of generators comes in.

No other congruences have been studied from this point of view; the present paper aims to do for congruences of order 3 what Kummer did for those of order 2.*

I wish to express my appreciation and thanks to Professor Virgil Snyder for advice and assistance in the preparation of this paper.

1. The totality of all lines of space satisfying two conditions is called a *congruence.* The *order* of a *congruence* is the number of lines of the congruence which pass through any point of space. The *class* of a *congruence* is the number of lines of the congruence which lie in any plane in space. A *singular point* is a point through which pass an infinite number of lines of the congruence. A *singular plane* is a plane in which lie an infinite number of lines of the congruence. A *basis point* of a system of quadrics is a point common to every quadric of the system.

2. Congruences of certain types may be arranged on a system of quadric surfaces. Such a congruence arises if the two conditions which determine it are of the form:

$$\Sigma a_i x + \sigma \Sigma b_i x = 0,$$
$$\Sigma \bar{a}_i x + \sigma \Sigma \bar{b}_i x = 0,$$

where a_i, $\bar{a}_i$, b_i, $\bar{b}_i$ are functions of a parameter λ, and the summation sign indicates that the a_i, $\bar{a}_i$, b_i, $\bar{b}_i$ are to be taken with the homogeneous point coordinates x, y, z, w. Solving each equation for σ:

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* Since this paper was written, R. Baldus, in an Erlangen dissertation, 1910, entitled "Ueber Strahlensysteme, welche unendlich viele Regelflächen zweiten Grades enthalten," has contributed some interesting information which supplements certain parts of this paper.

$$\frac{\Sigma a_i x}{\Sigma b_i x}=\frac{\Sigma \bar{a}_i x}{\Sigma \bar{b}_i x}=\sigma. \tag{1}$$

In this form it is evident that the lines for all values of σ define the generators of one system of the family of quadrics:

$$H(\lambda)\equiv(\Sigma a_i x)(\Sigma \bar{b}_i x)-(\Sigma \bar{a}_i x)(\Sigma b_i x)=0. \tag{2}$$

If $\Sigma \bar{a}_i x$ and $\Sigma b_i x$ be interchanged in the equations, there results a new congruence,

$$\frac{\Sigma a_i x}{\Sigma \bar{a}_i x}=\frac{\Sigma b_i x}{\Sigma \bar{b}_i x}=\tau, \tag{3}$$

the lines of which define the generators of the second system in the same family of quadrics:

$$H(\lambda)=0. \tag{2}$$

Two such congruences will be referred to as the σ and τ congruences respectively, and will be called *conjugate congruences.* The envelop of the system of quadrics $H(\lambda)$ is the *focal surface* of the conjugate congruences σ and τ.

3. In this paper will be discussed the congruences defined by the equations

$$\text{(I)}\begin{cases}\dfrac{a+b\lambda+c\lambda^2}{x+y\lambda}=\dfrac{a'+b'\lambda+c'\lambda^2}{z+w\lambda}=\sigma, & (4)\\[2ex] \dfrac{a+b\lambda+c\lambda^2}{a'+b'\lambda+c'\lambda^2}=\dfrac{x+y\lambda}{z+w\lambda}=\tau, & (5)\end{cases}$$

$$\text{(II)}\begin{cases}\dfrac{a+b\lambda+c\lambda^2+d\lambda^3}{x+y\lambda+z\lambda^2}=\dfrac{a'+b'\lambda}{w}=\sigma, & (6)\\[2ex] \dfrac{a+b\lambda+c\lambda^2+d\lambda^3}{a'+b'\lambda}=\dfrac{x+y\lambda+z\lambda^2}{w}=\tau, & (7)\end{cases}$$

in which a, b, c, d, a', b', c' are linear functions of the variables x, y, z, w; and λ, σ and τ are variable parameters. The congruences will be discussed both in case the linear functions are general and in case they satisfy certain given conditions. It is to be observed that both parameters, λ, σ (τ), enter rationally and one of them, σ (τ), linearly in the equation. The conjugate congruences (4) and (5) can be arranged on the family of quadrics:

$$H(\lambda)\equiv(a+b\lambda+c\lambda^2)(z+w\lambda)-(a'+b'\lambda+c'\lambda^2)(x+y\lambda)=0. \tag{8}$$

Each congruence is composed of the generators of one system on the family of quadrics. Similarly, the conjugate congruences (6) and (7) are composed of the two sets of generators of the family:

$$H'(\lambda)\equiv w(a+b\lambda+c\lambda^2+d\lambda^3)-(a'+b'\lambda)(x+y\lambda+z\lambda^2)=0. \tag{9}$$

The two systems of generators of each family of quadrics are rationally separable.

4. If the coordinates of a fixed point be substituted in equation (8) [or (9)], it is evident that there exist three values of λ satisfying the equation; *i. e.*, through every point of space pass three quadrics of the family.

Through every point of space pass three lines of the σ and τ conqruences, respectively; hence the congruence (σ or τ) is of order 3.

5. To determine the class of a congruence, impose the condition that an arbitrary plane,

$$A_1x+A_2y+A_3z+A_4w=0,$$

shall be tangent to a quadric of $H(\lambda)=0$. This will give an equation in λ satisfied by nine values:

$$\begin{vmatrix} \frac{\partial^2 H}{\partial x^2} & \frac{\partial^2 H}{\partial x\partial y} & \frac{\partial^2 H}{\partial x\partial z} & \frac{\partial^2 H}{\partial x\partial w} & A_1 \\ \frac{\partial^2 H}{\partial x\partial y} & \frac{\partial^2 H}{\partial y^2} & \frac{\partial^2 H}{\partial y\partial z} & \frac{\partial^2 H}{\partial y\partial w} & A_2 \\ \frac{\partial^2 H}{\partial x\partial z} & \frac{\partial^2 H}{\partial y\partial z} & \frac{\partial^2 H}{\partial z^2} & \frac{\partial^2 H}{\partial z\partial w} & A_3 \\ \frac{\partial^2 H}{\partial x\partial w} & \frac{\partial^2 H}{\partial y\partial w} & \frac{\partial^2 H}{\partial z\partial w} & \frac{\partial^2 H}{\partial w^2} & A_4 \\ A_1 & A_2 & A_3 & A_4 & 0 \end{vmatrix} \equiv f_9(\lambda)=0. \qquad (10)$$

In any plane of space lie three lines of the σ and τ congruences respectively; hence the congruence (σ or τ) is of class 9.

6. Corresponding to those values of λ for which the discriminant of the quadric H vanishes, the quadric becomes a cone. This discriminant is of order 12 in λ, $D_{12}(\lambda)$.

The necessary and sufficient condition* that the systems of generators depending rationally on a parameter λ shall be separable into two conjugate congruences is that the discriminant shall be a perfect square in λ. Hence we have

$$D_{12}(\lambda)=[f_6(\lambda)]^2; \qquad (11)$$

thus the discriminant possesses six double roots and there exists in H a cone for each such root.

The general congruences σ and τ are arranged on a family of quadrics which includes six quadric cones. The lines of the cones belong to both the σ and τ congruences.

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* Snyder: "Conjugate Line Congruences Contained in a Bundle of Quadric Surfaces," *Transactions Amer. Math. Soc.*, Vol. XI (1910), pp. 371–387.

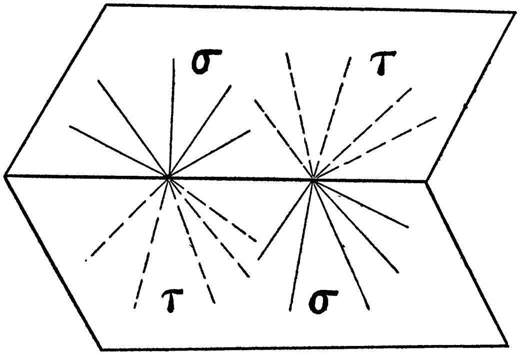
7. A singular plane may arise either from a basis plane or from the breaking into planes of one of the cones included in H, while a singular point may arise from a basis point or the vertex of a cone in H.

Imposing the condition that one of the cones of the system shall degenerate to a pair of planes, *i. e.*, making the minors of the Hessian of $H(\lambda)=0$ vanish for one value λ_1 of λ, the $f_9(\lambda)=0$ which determines the class of the congruence will have that value λ_1 as a factor, and the remaining function will be of order 8, $F_8(\lambda)=0$. This shows that the class of the congruence is lowered by one unit by the presence of a degenerate cone in the system. The argument may be repeated for each additional cone.

The class of the congruence σ (τ) is diminished by unity by the degeneracy of each of the six quadric cones into a pair of planes.

8. Suppose in equation (1) that $\Sigma a_i x \equiv (\lambda-\lambda_1)\Sigma A_i x$, but that no other Σ contains $(\lambda-\lambda_1)$ as a factor. Then, for $\lambda=\lambda_1$, the equation becomes:

$$\frac{0\cdot\Sigma A_i x}{\Sigma\bar{A}_i x}=\frac{\Sigma B_i x}{\Sigma\bar{B}_i x}=\tau, \tag{12}$$



where $A_i\equiv a_i$ for $\lambda=\lambda_1$, etc. Since τ is a parameter, taking all values, it follows that $(\Sigma\bar{A}_i x)\tau=0\cdot\Sigma A_i x$ requires that $\Sigma\bar{A}_i x=0$ contain a line of the τ congruence for every value of τ. Hence $\Sigma\bar{A}_i x=0$ is a singular plane. Further, any line of the τ congruence in this plane passes through the point $\Sigma B_i x=0$, $\Sigma\bar{B}_i x=0$, $\Sigma\bar{A}_i x=0$. This point will be called the "τ vertex" in the plane $\Sigma\bar{A}_i x=0$. The plane $\Sigma A_i x=0$ is likewise a singular plane for the σ congruence, all lines of which in the plane pass through the point $\Sigma\bar{A}_i x=0$, $\Sigma A_i x=0$, $\Sigma B_i x=0$. This is the "σ vertex" in the plane $\Sigma\bar{A}_i x=0$. Similarly, for $\lambda=\lambda_1$, $\Sigma B_i x_i=0$ is a singular plane with τ vertex at $\Sigma\bar{A}_i x=0$, $\Sigma B_i x=0$, $\Sigma A_i x=0$, and the σ vertex at the point $\Sigma\bar{A}_i x=0$, $\Sigma B_i x=0$, $\Sigma\bar{B}_i x=0$. Since λ_1 is any root of all the first minors of the Hessian, the above argument holds for any cone included in H which breaks into two planes. Hence the following theorem holds:

When a quadric of $H(\lambda)$ breaks up into a pair of planes, the lines of each system of generators become pencils of lines, one pencil in each plane at each of two vertices, which are in general distinct, lying on the line of intersection of the planes. The σ vertex in one is the τ vertex in the other, and vice versa.

9. For any basis point P_1 of a cubic congruence all the lines are given by the equations

$$H(\lambda)\equiv H_0\lambda^3+H_1\lambda^2+H_2\lambda+H_3=0,$$

and the tangent plane

$$\Sigma x\frac{\partial H_0}{\partial x_1}\lambda^3+\ldots\ldots=0.$$

The λ eliminant is therefore of degree 9; *i. e.*, the lines of both σ and τ congruences form a cone of order 9. If the σ lines and τ lines form distinct cones, the sum of the orders of such cones is 9. Further, if the class of the congruence be decreased by unity by the breaking up of a cone into a pair of planes, there will appear a factor common to the two equations, so that the degree of the eliminant is decreased by one.

The sum of the orders of the singular cones of all σ and τ lines through a basis point is always equal to the class of each congruence.

All of the above statements hold for either the conjugate congruences (4) and (5) or the conjugate congruences (6) and (7). Hereafter the congruence determined from (4) and (5) by imposing various conditions will be referred to as Type I; those determined from (6) and (7) as Type II. In general the σ congruence will be discussed, since the properties of the conjugate τ are immediately deducible from those of the σ. When there is danger of ambiguity, both congruences are taken into consideration. The notation $\mathrm{I}(3,k)^{a,b}_{c,d}$ stands for a congruence of Type I, order 3, class k, with a basis points, with b basis planes, with c singular points which are vertices of cones or pencils in degenerate cones, with d singular planes arising from a cone breaking into pairs of planes.

10. Since the focal surface of a congruence has been defined as the envelop of the family of quadrics $H(\lambda)=0$, the focal surface of the rational congruence $(3,9)$ is of order 8. The vertices of the six cones of the family will be double points on it, since they result from the double roots of the discriminant of $H(\lambda)=0$. Each quadric of the family H will intersect the focal surface in a curve C_8 and the curve of contact C_4, defined by the equations

$$H(\lambda)=0,\ \frac{\partial H(\lambda)}{\partial\lambda}=0.$$

Since there exists a parameter λ in these equations, the focal surface contains

∞^1 twisted quartics of the first kind. The six cones of the family H will each touch the focal surface in a quartic curve, having a double point at the vertex of the cone. Further, the equation for

$$H(\lambda)\equiv H_0\lambda^3+3H_1\lambda^2+3H_2\lambda+H_3=0$$

will have a triple root if

$$\frac{H_0}{H_1}=\frac{H_1}{H_2}=\frac{H_2}{H_3}.$$

Hence the focal surface has the intersection of these two quartic surfaces as a cuspidal edge. This cuspidal edge is of order 12, since the intersection of $H_1=0$, $H_2=0$ does not lie on $H_0H_3=H_1H_2$.

Hence, the focal surface of the general σ (τ) *congruence contains* ∞^1 *twisted quartics of the first kind, and six quartic curves with one double point each. The double points of these six curves are the only double points on the surface. The surface contains in addition to these curves a cuspidal edge of order* 12.

11. Every line of the congruence will cut in two points the quartic curve of contact of its corresponding quadric and the focal surface F_8. Hence, every line of the congruence is a bitangent of the focal surface. The transformation S, defined by interchanging the two points of contact of a σ line (bitangent) of the focal surface, is an involutoric, non-linear, birational transformation under which the focal surface F_8 is invariant. Under this transformation, a point has its image always on the C_4 of contact which passes through this point. Similarly, the same surface F_8 is the focal surface of the conjugate congruence τ. The operation T of interchanging the points of contact of a τ line is also of order 2. The group generated by S and T is generally of infinite order. In all the congruences discussed in this paper, the order of the focal surface is the same as in the case of the general congruence, *i. e.*, it is of order 8.

TYPE I.

12. The general congruence

$$\frac{a+b\lambda+c\lambda^2}{x+y\lambda}=\frac{a'+b'\lambda+c'\lambda^2}{z+w\lambda}=\sigma \tag{4}$$

has no basis planes and no basis points. Hence it has no singular planes and only six singular points, namely, the vertices of the quadric cones in H.

The symbol for the conjugate congruences (4) *and* (5) *is* I$(3,9)_6$.

13. If the condition $a=0$ be imposed on (4) and (5), there result:

$$\frac{b\lambda+c\lambda^2}{x+y\lambda}=\frac{a'+b'\lambda+c'\lambda^2}{z+w\lambda}=\sigma,\quad \frac{b\lambda+c\lambda^2}{a'+b'\lambda+c'\lambda^2}=\frac{x+y\lambda}{z+w\lambda}=\tau. \tag{13}$$

For the value $\lambda=0$, the quadric of H is $a'x=0$, *i.e.*, a pair of planes $a'=0$, $x=0$. From Arts. 4, 7, 8 results: The congruence (13) has the singular planes $a'=0$, $x=0$, seven singular points, the vertices of five cones, and the σ vertices in the singular planes:

$$x=0,\ x=a'=z=0;\quad a'=0,\ x=a'=b=0.$$

The symbol for this congruence is $\mathrm{I}(3,\,8)_{7,2}$.

14. In a manner similar to the above, if on (13) be imposed the additional condition $c'=0$, a second cone breaks into plane pencils. The table for these planes and points includes those of Art. 13 and in addition:

$$c=0,\ c=w=b'=0;\quad w=0,\ w=c=y=0.$$

The equations for the congruence are

$$\frac{b\lambda+c\lambda^2}{x+y\lambda}=\frac{a'+b'\lambda}{z+w\lambda}=\sigma,\quad \frac{b\lambda+c\lambda^2}{a'+b'\lambda}=\frac{x+y\lambda}{z+w\lambda}=\tau. \tag{14}$$

The symbol for the congruence (14) *is* $\mathrm{I}(3,\,7)_{8,4}$.

15. Assuming[*] that $\lambda=1$ is to be a third value for which a cone degenerates, the necessary and sufficient condition to be imposed in addition to the preceding is:

$$b+c-k(a'+b')=p[(x+y)-k(z+w)].$$

Then

$$c=p[x+y-k(z+w)]-b+k(a'+b')=\bar{c}.$$

Substituting for c its value $\bar{c}$ and factoring:

$$\frac{(1-\lambda)[b\lambda-px(1+\lambda)-py\lambda+kpz(1+\lambda)+pkw\lambda-ka'\lambda+1-kb'\lambda]}{x+y\lambda-k(z+w\lambda)}$$
$$=\frac{a'+b'\lambda-p(z+w\lambda)}{z+w\lambda}=\sigma. \tag{15}$$

Applying Arts. 1, 7, 8, the complete statement of planes and vertices includes the planes and points of Art. 14 and in addition:

$$x+y-k(z+w)=0,\qquad x+y=a'+b'=z+w=0;$$
$$a'+b'-p(z+w)=0,\quad x+y-k(z+w)=a'+b'-p(z+w)$$
$$=b-p(2x+y-kz)-ka'=0.$$

The congruence (15) *has the symbol* $\mathrm{I}(3,\,6)_{9,6}$.

16. In order that a fourth cone for $\lambda=\lambda_1$ shall break into planes, the necessary and sufficient condition to be imposed on (15) is:

$$b\lambda_1+c\lambda_1^2-l(\bar{a}'+b'\lambda_1)=r[x+y\lambda_1-l(z+w)\lambda_1],$$

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[*] It is no restriction to impose these three values of λ for the parameters of three composite cones.

whence

$$b=\frac{r[x+y\lambda_1-l(z+w\lambda_1)]-\bar{c}\lambda_1^2+l(a'+b'\lambda)}{\lambda_1}=\bar{b}.$$

Substituting, subtracting s from each member and factoring:

$$\frac{(\lambda-\lambda_1)\{rx-lsz+\bar{c}\lambda\lambda_1+la'\}}{\lambda_1[x+y\lambda-l(z+w\lambda)]}=\frac{a'+b'\lambda-r(z+w\lambda)}{z+w\lambda}=\sigma. \tag{16}$$

From Arts. 7, 8, the complete statement of planes and points includes those of Art. 15, and in addition:

$$\begin{aligned}&(x+y\lambda_1)-l(z+w\lambda_1)=0, && a'+b'\lambda_1=z+w\lambda_1=x+y\lambda_1=0;\\ &(a'+b'\lambda_1)-r(z+w\lambda_1)=0, && x+y\lambda_1-l(z+w\lambda_1)=a'+b'\lambda_1-r(z+w\lambda_1)\\ &&&=r(x-lz)+\bar{c}\lambda_1^2+la'=0.\end{aligned}$$

The symbol for the congruence (16) *is* $\mathrm{I}(3,5)_{10,8}$.

17. That the fifth cone degenerate, the necessary and sufficient condition on (16) is:

$$\bar{b}\lambda_2+\bar{c}\lambda_2^2-m(a'+b'\lambda_2)=s[x+y\lambda_2-m(z+w\lambda_2)],$$

whence

$$a'=\frac{-s[x+y\lambda_2-m(z+w\lambda_2)]-b'\lambda_2 m-\bar{b}\lambda_2-\bar{c}\lambda_2^2}{m}=\bar{a}'.$$

Substituting, subtracting s from each member and factoring:

$$\frac{(\lambda-\lambda_2)[\bar{b}+\bar{c}(\lambda+\lambda_2)-sy-smw-b'm]}{x+y\lambda-m(z+w\lambda)}=\frac{\bar{a}'+b'\lambda-s(z+w\lambda)}{z+w\lambda}=\sigma. \tag{17}$$

The complete statement of planes and points includes those of Art. 16 and in addition:

$$\begin{aligned}&x+y\lambda_2-m(z+w\lambda_2)=0, && \bar{a}'+b'\lambda_2=x+y\lambda_2=z+w\lambda_2=0;\\ &\bar{a}'+b'\lambda_2-s(z+w\lambda_2)=0, && x+y\lambda_2-m(z+w\lambda_2)=\bar{a}'+b'\lambda_2-s(z+w\lambda_2)\\ &&&=\bar{b}+2\bar{c}\lambda_2-sy+smw-b'm=0.\end{aligned}$$

The symbol for the congruence (17) *is* $\mathrm{I}(3,4)_{11,10}$.

18. In order that all six of the quadric cones of H may be composed of pairs of planes, the following six conditions must be simultaneously imposed on $H(\lambda)=0$.

$$\begin{gathered}a=0,\quad c'=0,\\ b+c-k(a'+b')=p[x+y-k(z+w)],\\ b\lambda_1+c\lambda_1^2-l(a'+b'\lambda_1)=r[x+y\lambda_1-l(z+w\lambda_1)],\\ b\lambda_2+c\lambda_2^2-m(a'+b'\lambda_2)=s[x+y\lambda_2-m(z+w\lambda_2)],\\ b\lambda_3+c\lambda_3^2-n(a'+b'\lambda_3)=t[x+y\lambda_3-n(z+w\lambda_3)].\end{gathered}$$

Imposing these conditions on $H(\lambda)=0$ gives

$$\begin{vmatrix} 1 & 1 & k & k & p[x+y-k(z+w)] \\ \lambda_1 & \lambda_1^2 & l & l\lambda_1 & r[x+y\lambda_1-l(z+w\lambda_1)] \\ \lambda_2 & \lambda_2^2 & m & m\lambda_2 & s[x+y\lambda_2-m(z+w\lambda_2)] \\ \lambda_3 & \lambda_3^2 & n & n\lambda_3 & t[x+y\lambda_3-n(z+w\lambda_3)] \\ z+w\lambda & \lambda(z+w\lambda) & x+y\lambda & \lambda(x+y\lambda) & 0 \end{vmatrix}=0. \qquad (18)$$

Reducing this determinant, we have:

$$\frac{\lambda\begin{vmatrix} 1-\lambda & k & k & p[x+y-k(z+w)] \\ \lambda_1(\lambda_1-\lambda) & l & l\lambda_1 & r[x+y\lambda_1-l(z+w\lambda_1)] \\ \lambda_2(\lambda_2-\lambda) & m & m\lambda_2 & s[x+y\lambda_2-m(z+w\lambda_2)] \\ \lambda_3(\lambda_3-\lambda) & n & n\lambda_3 & t[x+y\lambda_3-n(z+w\lambda_3)] \end{vmatrix}}{x+y\lambda}$$

$$=\frac{-\begin{vmatrix} 1 & 1 & k(1-\lambda) & p[x+y-k(z+w)] \\ \lambda_1 & \lambda_1^2 & l(\lambda_1-\lambda) & r[x+y\lambda_1-l(z+w\lambda_1)] \\ \lambda_2 & \lambda_2^2 & m(\lambda_2-\lambda) & s[x+y\lambda_2-m(z+w\lambda_2)] \\ \lambda_3 & \lambda_3^2 & n(\lambda_3-\lambda) & t[x+y\lambda_3-n(z+w\lambda_3)] \end{vmatrix}}{z+w\lambda}=\sigma$$

$$\equiv\frac{S}{x+y\lambda}=\frac{T}{z+w\lambda}=\sigma. \qquad (19)$$

This equation is in the exact form of the general equation for congruences of Type I. By reference to the conditions imposed and Art. 8 it is possible to write out the complete table of singular planes and points for the six values of λ: $0, \infty, 1, \lambda_1, \lambda_2, \lambda_3$. This table is as follows. The notation $(1, a)$ indicates the first singular plane for value $\lambda=a$. $(\sigma, 1, a)$ indicates the σ vertex in the plane $(1, a)$.

$$(1, 0)\equiv 0=x; \quad (\sigma, 1, 0): x=z=\begin{vmatrix} 1 & k & k & p(y-wk) \\ \lambda_1^2 & l & l\lambda_1 & r\lambda_1(y-wl) \\ \lambda_2^2 & m & m\lambda_2 & s\lambda_2(y-wm) \\ \lambda_3^2 & n & n\lambda_3 & t\lambda_3(y-wn) \end{vmatrix}=0.$$

Let A, B, C, D indicate the $1, \ldots, 4$ terms of the last column of the determinant in (19).

$$(2, 0)\equiv\begin{vmatrix} 1 & 1 & k & A \\ \lambda_1^2 & \lambda_1 & l\lambda_1 & B \\ \lambda_2^2 & \lambda_2 & m\lambda_2 & C \\ \lambda_3^2 & \lambda_3 & n\lambda_3 & D \end{vmatrix}=0; \quad (\sigma, 2, 0,): (1, 0)=(2, 0)=\begin{vmatrix} 1 & k & k & A \\ \lambda_1^2 & l & l\lambda_1 & B \\ \lambda_2^2 & m & m\lambda_2 & C \\ \lambda_3^2 & n & n\lambda_3 & D \end{vmatrix}=0.$$

$$(1, \infty)\equiv w=0; \quad (\sigma, 1, \infty): (1, \infty)=(2, \infty)=y=0.$$

$$(2,\infty)\equiv\begin{vmatrix}1 & k & k & A\\ \lambda_1 & l & l\lambda_1 & B\\ \lambda_2 & m & m\lambda_2 & C\\ \lambda_3 & n & n\lambda_3 & D\end{vmatrix}=0;\quad (\sigma, 2, \infty):(1,\infty)=(2,\infty)=\begin{vmatrix}1 & 1 & k & A\\ \lambda_1 & \lambda_1^2 & l & B\\ \lambda_2 & \lambda_2^2 & m & C\\ \lambda_3 & \lambda_3^2 & n & D\end{vmatrix}=0.$$

$$(1,1)\equiv x+y-k(z+w)=0;\quad (\sigma, 1, 1):(1,1)=(1,2)=z+w=0.$$

$$(2,1)\equiv\begin{vmatrix}1 & 1 & 0 & A\\ \lambda_1 & \lambda_1^2 & l(\lambda_1-1) & B\\ \lambda_2 & \lambda_2^2 & m(\lambda_2-1) & C\\ \lambda_3 & \lambda_3^2 & n(\lambda_3-1) & D\end{vmatrix}=0;$$

$$(\sigma, 2, 1,):(1,1)=(1,2)=\left.\frac{S-kT-p[x+y-kz-kw]}{1-\lambda}\right]_{\lambda\doteq 1}=0.$$

$$(1,\lambda_1)\equiv x+y\lambda_1-l(z+w\lambda_1)=0;\quad (\sigma, 1, \lambda_1):x+y\lambda_1=z+w\lambda_1=(1,\lambda_2)=0.$$

$$(2,\lambda_1)\equiv\begin{vmatrix}1 & 1 & k & A\\ \lambda_1 & \lambda_1^2 & 0 & B\\ \lambda_2 & \lambda_2^2 & m(\lambda_2-\lambda_1) & C\\ \lambda_3 & \lambda_3^2 & n(\lambda_3-\lambda_1) & D\end{vmatrix}=0;$$

$$(\sigma, 2, \lambda_1):(1,\lambda_1)=(2,\lambda_1)=\left.\frac{S-lT-r[x+y\lambda_1-l(z+w\lambda_1)]}{\lambda-\lambda_1}\right]_{\lambda\doteq\lambda_1}=0.$$

$$(1,\lambda_2)\equiv x+y\lambda_2-m(z+w\lambda_2)=0;\quad (\sigma, 1, \lambda_2):x+y\lambda_2=z+w\lambda_2=(2,\lambda_2)=0.$$

$$(2,\lambda_2)\equiv\begin{vmatrix}1 & 1 & k & A\\ \lambda_1 & \lambda_1^2 & l(\lambda_1-\lambda_2) & B\\ \lambda_2 & \lambda_2^2 & 0 & C\\ \lambda_3 & \lambda_3^2 & n(\lambda_3-\lambda_2) & D\end{vmatrix}=0;$$

$$(\sigma, 2, \lambda_2):(1,\lambda_2)=(2,\lambda_2)=\left.\frac{S-mT-s[x+y\lambda_2-m(z+w\lambda_2)]}{\lambda-\lambda_2}\right]_{\lambda\doteq\lambda_2}=0.$$

$$(1,\lambda_3)\equiv x+y\lambda_3-n(z+w\lambda_3)=0;\quad (\sigma, 1, \lambda_3):x+y\lambda_3=z+w\lambda_3=(2,\lambda_3)=0.$$

$$(2,\lambda_3)\equiv\begin{vmatrix}1 & 1 & k & A\\ \lambda_1 & \lambda_1^2 & l(\lambda_1-\lambda_3) & B\\ \lambda_2 & \lambda_2^2 & m(\lambda_2-\lambda_3) & C\\ \lambda_3 & \lambda_3^2 & 0 & D\end{vmatrix}=0;$$

$$(\sigma, 2, \lambda_3):(1,\lambda_3)=(2,\lambda_3)=\left.\frac{S-nT-t[x+y\lambda_3-n(z+w\lambda_3)]}{\lambda-\lambda_3}\right]_{\lambda\doteq\lambda_3}=0.$$

The symbol for the congruence (18) *is* $\mathrm{I}(3,3)_{12,12}$.

19. In the above work care has been taken that no basis point, line, or plane should be allowed to enter. By imposing conditions on the eight arbitrary constants k, l, m, n, p, r, s, t it is possible to introduce one, two, three

or four basis points, thereby obtaining the four pairs of congruences: $I(3,3)^{1,1}_{12,12}$; $I(3,3)^{2,2}_{12,12}$; $I(3,3)^{3,3}_{12,12}$; $I(3,3)^{4,4}_{12,12}$. The introduction of a basis point in a $(3,3)$ simultaneously brings in a basis plane through the basis point.

The case of three basis points was obtained synthetically by Roccella* and is also mentioned by Snyder.† The case of four basis points and four basis planes is mentioned by Fano‡ and Montesano.§ It is a particular case of Roccella's configuration and also of that of Hirst.|| C. Corrona¶ seems to have been familiar with it, but the configuration of points and planes is not discussed.

The four basis points and the necessary and sufficient conditions for each are as follows, O_i being the basis point; A, B, C, D as in Art. 18; and I, II, III, IV the following determinants:

$$\mathrm{I}=\begin{vmatrix} 1 & k & k & A \\ \lambda_1 & l & l\lambda_1 & B \\ \lambda_2 & m & m\lambda_2 & C \\ \lambda_3 & n & n\lambda_3 & D \end{vmatrix}; \quad \mathrm{II}=\begin{vmatrix} 1 & k & k & A \\ \lambda_1^2 & l & l\lambda_1 & B \\ \lambda_2^2 & m & m\lambda_2 & C \\ \lambda_3^2 & n & n\lambda_3 & D \end{vmatrix};$$

$$\mathrm{III}=\begin{vmatrix} 1 & 1 & k & A \\ \lambda_1 & \lambda_1^2 & l & B \\ \lambda_2 & \lambda_2^2 & m & C \\ \lambda_3 & \lambda_3^2 & n & D \end{vmatrix}; \quad \mathrm{IV}=\begin{vmatrix} 1 & 1 & k & A \\ \lambda_1 & \lambda_1^2 & l\lambda_1 & B \\ \lambda_2 & \lambda_2^2 & m\lambda_2 & C \\ \lambda_3 & \lambda_3^2 & n\lambda_3 & D \end{vmatrix}.$$

$$O_1 : w=\mathrm{IV}=x+y\lambda_2-mz=0.$$
$$O_2 : x=\mathrm{I}=y-k(z+w)=0.$$
$$O_3 : w=\mathrm{IV}=x+y-kz=0.$$
$$O_4 : x=\mathrm{I}=y\lambda_1-l(z+w\lambda_1)=0.$$

If A_i, B_i, C_i, D_i indicate that the conditions of O_i are imposed, the conditions for O_1 are:

$$-\frac{z}{y}\begin{vmatrix} 1 & k & k & A_1 \\ \lambda_1 & l & l\lambda_1 & B_1 \\ \lambda_2 & m & m\lambda_2 & 0 \\ \lambda_3 & n & n\lambda_3 & D_1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & k & A_1 \\ \lambda_1 & \lambda_1^2 & l & B_1 \\ \lambda_2 & \lambda_2^2 & m & 0 \\ \lambda_3 & \lambda_3^2 & n & D_1 \end{vmatrix} = \frac{z}{x}\begin{vmatrix} 1 & k & k & A_1 \\ \lambda_1^2 & l & l\lambda_1 & B_1 \\ \lambda_2^2 & m & m\lambda_2 & 0 \\ \lambda_3^2 & n & n\lambda_3 & D_1 \end{vmatrix}.$$

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* Roccella: "Sugli enti geometrici dello spazio di rette," Piazza Armerina (1882).

† Snyder: "Surfaces Invariant under Infinite Discontinuous Birational Groups Defined by Line Congruences," American Journal of Mathematics, Vol. XXXII (1910), pp. 177–185.

‡ Fano: "Sopra alcune particolari congruenze di rette del 3° ordine," *Atti di Torino*, Vol. XXXVI (1900–1901), pp. 222–236.

§ Montesano: "Su di un complesso di rette di terzo grado," *Mem. di Bologno*, Ser. 5, Vol. III (1893), pp. 549–577.

|| Hirst: "On Cremonian Congruences," *Proceedings of the London Mathematical Society*, Vol. XIV (1883), pp. 259–301.

¶ Corrona: "Sopra un complesso di rette di quarto grado," *Rend. di Palermo*, Vol. XVI (1902), pp. 359–375.

The conditions for O_2 are:

$$\frac{y}{w}\begin{vmatrix} 1 & 1 & k & 0 \\ \lambda_1 & \lambda_1^2 & l & B_2 \\ \lambda_2 & \lambda_2^2 & m & C_2 \\ \lambda_3 & \lambda_3^2 & n & D_2 \end{vmatrix} = \begin{vmatrix} 1 & k & k & 0 \\ \lambda_1^2 & l & l\lambda_1 & B_2 \\ \lambda_2^2 & m & m\lambda_2 & C_2 \\ \lambda_3^2 & n & n\lambda_3 & D_2 \end{vmatrix} = -\frac{y}{z}\begin{vmatrix} 1 & 1 & k & 0 \\ \lambda_1 & \lambda_1^2 & l\lambda_1 & B_2 \\ \lambda_2 & \lambda_2^2 & m\lambda_2 & C_2 \\ \lambda_3 & \lambda_3^2 & n\lambda_3 & D_2 \end{vmatrix}.$$

The conditions for O_3 are:

$$-\frac{z}{y}\begin{vmatrix} 1 & k & k & A_3 \\ \lambda_1 & l & l\lambda_1 & 0 \\ \lambda_2 & m & m\lambda_2 & C_3 \\ \lambda_3 & n & n\lambda_3 & D_3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & k & A_3 \\ \lambda_1 & \lambda_1^2 & l & 0 \\ \lambda_2 & \lambda_2^2 & m & C_3 \\ \lambda_3 & \lambda_3^2 & n & D_3 \end{vmatrix} = \frac{z}{w}\begin{vmatrix} 1 & k & k & A_3 \\ \lambda_1 & l & l\lambda_1 & 0 \\ \lambda_2 & m & m\lambda_2 & C_3 \\ \lambda_3 & n & n\lambda_3 & D_3 \end{vmatrix}.$$

The conditions for O_4 are:

$$\frac{y}{w}\begin{vmatrix} 1 & 1 & k & A_4 \\ \lambda_1 & \lambda_1^2 & l & 0 \\ \lambda_2 & \lambda_2^2 & m & C_4 \\ \lambda_3 & \lambda_3^2 & n & D_4 \end{vmatrix} = \begin{vmatrix} 1 & k & k & A_4 \\ \lambda_1^2 & l & l\lambda_1 & 0 \\ \lambda_2^2 & m & m\lambda_2 & C_4 \\ \lambda_3^2 & n & n\lambda_3 & D_4 \end{vmatrix} = -\frac{y}{z}\begin{vmatrix} 1 & 1 & k & A_4 \\ \lambda_1 & \lambda_1^2 & l\lambda_1 & 0 \\ \lambda_2 & \lambda_2^2 & m\lambda_2 & C_4 \\ \lambda_3 & \lambda_3^2 & n\lambda_3 & D_4 \end{vmatrix}.$$

Through O_1 pass planes $(1, \lambda_2) = (1, \infty) = (1, \lambda_3) = (2, \lambda_1) = (2, 0) = (2, 1) = 0$.
Through O_2: $(1, \lambda_2) = (1, 1) = (1, 0) = (2, \lambda_1) = (2, \lambda_3) = (2, \infty) = 0$.
Through O_3: $(1, \infty) = (1, 1) = (1, \lambda_1) = (2, 0) = (2, \lambda_3) = (2, \lambda_2) = 0$.
Through O_4: $(1, 0) = (1, \lambda_1) = (1, \lambda_3) = (2, \infty) = (2, 1) = (2, \lambda_2) = 0$.

With the four basis points are associated four basis planes indicated by Π_i, where O_i lies in Π_i, $i = 1, 2, 3, 4$. The σ vertices lie by sixes in the planes as follows:

In Π_1 lie $(\sigma, 2, \lambda_1)$; $(\sigma, 1, \lambda_2)$; $(\sigma, 2, 1)$; $(\sigma, 1, \lambda_3)$; $(\sigma, 2, 0)$; $(\sigma, 1, \infty)$.
In Π_2 lie $(\sigma, 1, \lambda_2)$; $(\sigma, 2, \lambda_3)$; $(\sigma, 1, 1)$; $(\sigma, 2, \infty)$; $(\sigma, 1, 0)$; $(\sigma, 2, \lambda_1)$.
In Π_3 lie $(\sigma, 2, \lambda_3)$; $(\sigma, 1, 1)$; $(\sigma, 2, 0)$; $(\sigma, 1, \infty)$; $(\sigma, 1, \lambda_1)$; $(\sigma, 2, \lambda_2)$.
In Π_4 lie $(\sigma, 2, \lambda_2)$; $(\sigma, 1, \lambda_1)$; $(\sigma, 1, \lambda_3)$; $(\sigma, 2, 1)$; $(\sigma, 1, 0)$; $(\sigma, 2, \infty)$.

Each basis plane contains six vertices of pencils and one basis point. Dually, each basis point lies in six planes of pencils and in one basis plane. Each vertex of a pencil lies in two basis planes, and in two planes of pencils. Dually, each plane of pencils contains two basis points and two vertices of pencils.

20. This configuration of sixteen points and sixteen planes contains four basis points O_i; four basis planes Π_i; twelve vertices of pencils P_{ik}, Q_{ik}; twelve planes of pencils σ_{ik}, τ_{ik}, where $i=1, \ldots, 4$, $k=1, \ldots, 4$, $i \neq k$ but $ik=ki$. In Π_i lie O_i and six points P_{ik}, Q_{ik}; Π_i and Π_k meet in l_{ik}, which contains P_{ik}, Q_{jl}, $i \neq k \neq j \neq l$; O_i, O_k, P_{ik} determine σ_{ik} and O_j, O_l, Q_{jl} determine τ_{ik};

σ_{ik} and τ_{ik} form a pair of singular planes which compose a quadric cone of the system H.

21. From Art. 9 it is evident that the sum of the orders of cones of all σ and τ lines through the basis points is three. It is easily seen that the σ lines form a plane pencil in each basis plane with vertex at the basis point. Hence the τ lines through the same point must form a quadric cone. Dual statements hold for the basis planes.

The σ lines form a pencil in each basis plane and through each basis point. The τ lines envelop a conic in each basis plane and define a quadric cone through each basis point.

22. If an attempt is made to arrange the basis points in the $\mathrm{I}\,(3,3)^{3,\,3}_{12,\,12}$ (Rocella's congruence) in a symmetrical manner, the planes of pencils arrange themselves in two distinct sets, six in each. Of the first set, each contains one and only one basis point, while of the second set each contains two basis points. To arrange them in this way requires an extra condition, leaving a fourth basis point impossible, unless six planes pass through a point, which will be shown later to be the case of eight basis points. The σ lines in the basis planes form pencils through the basis points. The τ lines envelop a conic in each basis plane and form a quadric cone through each basis point. The basis points for this case, together with the necessary and sufficient conditions, are:

O_1: $w=0$, $\mathrm{IV}=0$, $x+y\lambda_2-mz=0$. (Same as before for O_1.)
O_2: $x=0$, $\mathrm{I}=0$, $y-k(z+w)=0$. (Same as before for O_2.)
O_3: $\mathrm{I}=0$, $x+y\lambda_1-l(z+w\lambda_1)=0$, $x+y\lambda_3-n(z+w\lambda_3)=0$.

$$O_4:\ \begin{vmatrix} 1 & k & k & A_4 \\ \lambda_1^2 & l & l\lambda_1 & B_4 \\ \lambda_2^2 & m & m\lambda_2 & 0 \\ \lambda_3^2 & n & n\lambda_3 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 & k & A_4 \\ \lambda_1 & \lambda_1^2 & l & B_4 \\ \lambda_2 & \lambda_2^2 & m & 0 \\ \lambda_3 & \lambda_3^2 & n & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 & k & A_4 \\ \lambda_1 & \lambda_1^2 & l\lambda_1 & B_4 \\ \lambda_2 & \lambda_2^2 & m\lambda_2 & 0 \\ \lambda_3 & \lambda_3^2 & n\lambda_3 & 0 \end{vmatrix} = 0.$$

The focal surface* of the various forms of the Roccella congruences is self-dual.

TYPE II.

23. The general congruence

$$\frac{a+b\lambda+c\lambda^2+d\lambda^3}{x+y\lambda+z\lambda^2}=\frac{a'+b'\lambda}{w}=\sigma, \tag{6}$$

$$\frac{a+b\lambda+c\lambda^2+d\lambda^3}{a'+b'\lambda}=\frac{x+y\lambda+z\lambda^2}{w}=\tau \tag{7}$$

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* Cayley: "On a Sibi Reciprocal Surfaces," *Berlin Monatsberichte* (1878), pp. 309–313; *Coll. Papers*, Vol. X, pp. 252–255. Kummer: "Ueber diejenigen Flächen ....," *Berlin Monatsberichte* (1878), pp. 25–37.

has a basis point, $a'=b'=w=0$, and the basis plane $w=0$. To determine the σ configuration at this point, substitute the coördinates of the point in the equation (6); then

$$\frac{\bar{a}+\bar{b}\lambda+\bar{c}\lambda^2+\bar{d}\lambda^3}{\bar{x}+\bar{y}\lambda+\bar{z}\lambda^2}=\bar{\sigma}=\frac{\phi_3(\lambda)}{F_2(\lambda)},$$

where the $\bar{a}$, etc., indicate constant values when $a'=b'=w=0$. Substituting this value for σ in (6), the equations for all σ lines through the point are:

$$A_0\lambda^5+A_1\lambda^4+\ldots\ldots=0,$$
$$B_0\lambda^3+B_1\lambda^2+\ldots\ldots=0.$$

Eliminating λ, the resulting equation is of degree 8. Similar work for τ gives a plane pencil, which was to be expected, since the class of the congruence is 9.

The σ lines of the congruence (6) *through $a'=b'=w=0$ form a cone of order* 8, *while the τ lines of* (7) *through the same point form a plane pencil in the plane $w=0$.*

Since $w=0$ is a basis plane, containing a line of each quadric of the family H, it contains the vertices of the six quadric cones in the family H. The configuration of σ lines in $w=0$ is evident from equations (6). For $w=0$, $\sigma=\infty$; hence $x+y\lambda+z\lambda^2=0$, and the σ lines in the plane $w=0$ envelop the conic $x+y\lambda+z\lambda^2=0$. The orders of the σ and τ configurations in the basis plane $w=0$ are unchanged by the conditions imposed later.

The symbol for congruences (6), (7) *is* $\mathrm{II}(3,9)^{1,1}_{6,0}$.

25. If, in a manner similar to the work for Type I, conditions are imposed upon the equations (6) and (7) so that the cones degenerate one by one, the class of the congruence also is depressed and the order of the σ cone with vertex $a'=b'=w=0$ decreases by unity for each new condition.

In equations (6) let $a=0$; then

$$\frac{b\lambda+c\lambda^2+d\lambda^3}{x+y\lambda+z\lambda^2}=\frac{a'+b'\lambda}{w}=\sigma. \tag{20}$$

As before for $\lambda=0$, the quadric cone is composed of the planes $x=0$, $a'=0$. The σ vertices are:

$$(1',0)\equiv x=0,\quad (\sigma,1',0): x=w=a'=0;$$
$$(2',0)\equiv a=0,\quad (\sigma,2',0): x=b=a'=0.$$

The congruence has the basis plane $w=0$, the basis point $a'=b'=w=0$ and no additional basis points or basis planes.

The symbol for the congruence (20) *is* $\mathrm{II}(3,8)^{1,1}_{7,2}$.

26. Impose on (20) the new condition $d=0$. The new congruence,

$$\frac{b\lambda+c\lambda^2}{x+y\lambda+z\lambda^2}=\frac{a'+b'\lambda}{w}=\sigma, \tag{21}$$

has two cones, for $\lambda=0$, $\lambda=\infty$, which are composed of planes. The table of planes and vertices includes those of Art. 22 and

$$(1', \infty)\equiv z=0, \quad (\sigma, 1', \infty): z=b'=w=0;$$
$$(2', \infty)\equiv b=0, \quad (\sigma, 2', \infty): z=b'=c=0.$$

There are no new additional basis planes or basis points.

The symbol for the congruence (21) *is* $\mathrm{II}(3,7)^{1,1}_{8,4}$.

27. In order that three cones may be composed of planes, impose the condition that $\lambda-1$ may be a factor;* *i. e.*,

$$b+c=k(x+y+z). \tag{22}$$

$$\frac{[k(x+y+z)-c]\lambda+c\lambda^2}{x+y\lambda+z\lambda^2}=\frac{a'-b'\lambda}{w}=\sigma,$$

whence

$$\frac{(1-\lambda)[ky+kz(1+\lambda)-c\lambda]}{x+y\lambda+z\lambda^2}=\frac{a'+b'\lambda-kw}{w}=\sigma. \tag{23}$$

Then the pencils and planes are those of Art. 21 and the new ones:

$$(1', 1)\equiv x+y+z=0, \quad (\sigma, 1', 1): a'+b'=w=x+y+z=0;$$
$$(2', 1)\equiv a+b-kw=0, \quad (\sigma, 2', 1): x+y+z=a'+b'-kw=k(x-z)+c=0.$$

The symbol for this congruence is $\mathrm{II}(3,6)^{1,1}_{9,6}$.

28. In order that a fourth cone may be composed of planes, for $\lambda=\lambda_1$, the condition is

$$(1-\lambda_1)[ky+kz(1+\lambda_1)-c\lambda_1]=l(a'+b'\lambda_1-kw).$$

Then the congruence

$$\frac{(\lambda_1-\lambda)\left[(1-\lambda)(1-\lambda_1)(y+z)k-\dfrac{(a'-kw)(1+\lambda+\lambda_1)l+\lambda\lambda_1 lb'}{\lambda_1(1-\lambda_1)}\right]}{x+y\lambda+z\lambda^2}$$
$$=\frac{a'+b'\lambda-kw}{w}=\sigma \tag{24}$$

has four cones composed of planes. The complete table of planes and vertices of pencils includes those of Art. 24 and the following:

$$(1', \lambda_1)\equiv x+y\lambda_1+z\lambda_1^2-lw=0, \quad (\sigma, 1', \lambda_1): x+y\lambda_1+z\lambda_1^2=w=a'+b'\lambda_1=0;$$
$$(2', \lambda_1)\equiv a'+b'\lambda_1-kw=0, \quad (\sigma, 2', \lambda_1): x+y\lambda_1+z\lambda_1^2-lw=a'+b'\lambda_1-kw$$
$$=(1-\lambda_1)(1-\lambda_1)^2(y+z)k\lambda_1-(a'-kw)(1+\lambda_1+\lambda_1)l-\lambda_1\lambda_1 b'l=0.$$

The symbol for this congruence is $\mathrm{II}(3,5)^{1,1}_{10,8}$.

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* The choice of $0, \infty, 1$ as three vertices of cones is no restriction on the congruence.

29. The condition to be imposed on (23) that a fifth cone may be composed of planes is:

$$\begin{aligned}(\lambda_1-\lambda_2)\,[(1-\lambda_2)\,(1-\lambda_1)^2(y+z)k\lambda_1-(a'-kw)\,(1+\lambda_1+\lambda_2)l]\\ =m\lambda_1(1-\lambda_1)\,(a'+b'\lambda_2-kw)+(\lambda_1-\lambda_2)\lambda_1\lambda_2 lb'.\end{aligned}$$

Imposing this condition gives the congruence

$$\begin{aligned}&\frac{(\lambda_1-\lambda)\,[(1-\lambda)\,(1-\lambda_1)^2(y+z)k\lambda_1-(a'-kw)\,(1+\lambda+\lambda_1)l+\lambda\lambda_1 lb']}{\lambda_1(1-\lambda_1)\,(x+y\lambda+z\lambda^2)}\\ &=\frac{k(\lambda_1-\lambda_2)\,[wl(1+\lambda_1+\lambda_2)+(y+z)\,(1-\lambda_2)\,(1-\lambda_1)^2\lambda_1-\lambda_1\lambda_2 lb']-m\lambda_1(1-\lambda_1)b'(\lambda-\lambda_2)-kwl(1+\lambda_1+\lambda_2)}{w[l(1+\lambda_1+\lambda_2)+m\lambda_1(1-\lambda_1)]}\\ &\equiv\frac{A}{B(x+y\lambda+z\lambda^2)}=\frac{C}{Dw}=\sigma.\end{aligned} \tag{25}$$

The complete table of planes and vertices of pencils includes those of Art. 25 and in addition:

$$\begin{aligned}&(1',\lambda_2)\equiv x+y\lambda_2+z\lambda_2^2-mw=0,\ \ (\sigma,1',\lambda_2):\ x+y\lambda_2+z\lambda_2^2=w=a'+b'\lambda_2=0;\\ &(2',\lambda_2)\equiv a'+b'\lambda_2-kw=0,\ \ (\sigma,2',\lambda_2):\ x+y\lambda_2+z\lambda_2^2-mw=a'+b'\lambda_2-kw\\ &\qquad\qquad=\left.\frac{AD-mBC}{\lambda-\lambda_2}\right]_{\lambda\doteq\lambda_2}=0.\end{aligned}$$

The symbol for this congruence is $\mathrm{II}(3,4)^{1,1}_{11,10}$.

30. For the case in which all six cones are composed of pairs of planes, the six necessary and sufficient conditions to be simultaneously imposed are:

$$\begin{aligned}&a=0,\quad d=0,\quad b+c=k(x+y+z),\\ &b\lambda_1+c\lambda_1^2-l(x+y\lambda_1+z\lambda_1^2)=s(a'+b'\lambda_1-lw),\\ &b\lambda_2+c\lambda_2^2-m(x+y\lambda_2+z\lambda_2^2)=t(a'+b'\lambda_2-mw),\\ &b\lambda_3+c\lambda_3^2-n(x+y\lambda_3+z\lambda_3^2)=p(a'+b'\lambda_3-nw).\end{aligned}$$

Imposing these conditions on equation (9), there results:

$$\begin{vmatrix}1 & 1 & 0 & 0 & k(x+y+z)\\ \lambda_1 & \lambda_1^2 & s & s\lambda_1 & l(x+y\lambda_1-sw)\\ \lambda_2 & \lambda_2^2 & t & t\lambda_2 & m(x+y\lambda_2-tw)\\ \lambda_3 & \lambda_3^2 & p & p\lambda_3 & n(x+y\lambda_3-pw)\\ w\lambda & w\lambda^2 & x+y\lambda+z\lambda^2 & (x+y\lambda+z\lambda^2)\lambda & 0\end{vmatrix}=0.$$

Reducing this determinant to the form of (6) gives:

$$\frac{\lambda\begin{vmatrix}1-\lambda & 0 & 0 & k(x+y+z)\\ \lambda_1(\lambda_1-\lambda) & s & s\lambda_1 & l(x+y\lambda_1+z\lambda_1^2-sw)\\ \lambda_2(\lambda_2-\lambda) & t & t\lambda_2 & m(x+y\lambda_2+z\lambda_2^2-tw)\\ \lambda_3(\lambda_3-\lambda) & p & p\lambda_3 & n(x+y\lambda_3+z\lambda_3^2-pw)\end{vmatrix}}{x+y\lambda+z\lambda^2}$$

$$=\frac{-\begin{vmatrix}1 & 1 & 0 & k(x+y+z)\\ \lambda_1 & \lambda_1^2 & s(\lambda_1-\lambda) & l(x+y\lambda_1+z\lambda_1^2-sw)\\ \lambda_2 & \lambda_2^2 & t(\lambda_2-\lambda) & m(x+y\lambda_2+z\lambda_2^2-tw)\\ \lambda_3 & \lambda_3^2 & p(\lambda_3-\lambda) & n(x+y\lambda_3+z\lambda_3^2-pw)\end{vmatrix}}{w}=\sigma, \tag{26}$$

or

$$\frac{S}{x+y\lambda+z\lambda^2}=\frac{\tau}{w}=\sigma.$$

Applying Art. 8, and denoting by A, B, C, D the elements of the last column of these determinants, the complete table of planes and vertices of pencils is:

$$(1',0)\ :x=0;\quad (\sigma,1',0)\ :x=w=\begin{vmatrix}1&1&0&A\\ \lambda_1&\lambda_1^2&s\lambda_1&B\\ \lambda_2&\lambda_2^2&t\lambda_2&C\\ \lambda_3&\lambda_3^2&p\lambda_3&D\end{vmatrix}=0.$$

$$(2',0)\equiv\begin{vmatrix}1&1&0&A\\ \lambda_1&\lambda_1^2&s\lambda_1&B\\ \lambda_2&\lambda_2^2&t\lambda_2&C\\ \lambda_3&\lambda_3^2&p\lambda_3&D\end{vmatrix}=0;\quad (\sigma,2',0)\ :\ (1',0)=\begin{vmatrix}1&0&0&A\\ \lambda_1^2&s&s\lambda_1&B\\ \lambda_2^2&t&t\lambda_2&C\\ \lambda_3^2&p&p\lambda_3&D\end{vmatrix}=0=(2',0).$$

$$(1',\infty)\equiv z=0;\quad (\sigma,1',\infty)\ :\ (1',\infty)=(2',\infty)=w=0.$$

$$(2',\infty)\equiv\begin{vmatrix}1&1&0&A\\ \lambda_1&\lambda_1^2&s&B\\ \lambda_2&\lambda_2^2&t&C\\ \lambda_3&\lambda_3^2&p&D\end{vmatrix}=0;\quad (\sigma,2',\infty)\ :\ (1',\infty)=(2',\infty)=\begin{vmatrix}1&0&0&A\\ \lambda_1&s&s\lambda_1&B\\ \lambda_2&t&t\lambda_2&C\\ \lambda_3&p&p\lambda_3&D\end{vmatrix}=0.$$

$$(1',1)\equiv x+y+z=0;\quad (\sigma,1',1)\ :x+y+z=(2',1)=w=0.$$

$$(2',1)\equiv\begin{vmatrix}1&1&0&A\\ \lambda_1&\lambda_1^2&s(\lambda_1-1)&B\\ \lambda_2&\lambda_2^2&t(\lambda_2-1)&C\\ \lambda_3&\lambda_3^2&p(\lambda_3-1)&D\end{vmatrix}=0;\quad (\sigma,2',1)\ :\ (1',1)=(2',1)=\left.\frac{S+k}{\lambda-1}\right]_{\lambda\doteq 1}=0.$$

$$(1',\lambda_1)\equiv x+y\lambda_1+z\lambda_1^2-sw=0;\quad (\sigma,1',\lambda_1)\ :\ (1',\lambda_1)=(2',\lambda_1)=w=0.$$

$$(2',\lambda_1)\equiv\begin{vmatrix}1&1&0&A\\ \lambda_1&\lambda_1^2&0&B\\ \lambda_2&\lambda_2^2&t(\lambda_2-\lambda_1)&C\\ \lambda_3&\lambda_3^2&p(\lambda_3-\lambda_1)&D\end{vmatrix}=0;\ (\sigma,2',\lambda_1)\ :\ (1',\lambda_1)=(2',\lambda_1)=\left.\frac{S-sT}{\lambda-\lambda_1}\right]_{\lambda\doteq\lambda_1}=0.$$

$$(1',\lambda_2)\equiv x+y\lambda_2+z\lambda_2^2-tw=0;\quad (\sigma,1',\lambda_2)\ :\ (1',\lambda_2)=(2',\lambda_2)=w=0.$$

$$(2',\lambda_2)\equiv\begin{vmatrix}1&1&0&A\\ \lambda_1&\lambda_1^2&s(\lambda_1-\lambda_2)&B\\ \lambda_2&\lambda_2^2&0&C\\ \lambda_3&\lambda_3^2&p(\lambda_3-\lambda_2)&D\end{vmatrix}=0;\ (\sigma,2',\lambda_2)\ :\ (1',\lambda_2)=(2',\lambda_2)=\left.\frac{S-tT}{\lambda-\lambda_2}\right]_{\lambda=\lambda_2}=0.$$

$$(1',\lambda_3)\equiv x+y\lambda_3+z\lambda_3^2-pw=0;\quad (\sigma,1',\lambda_3)\ :\ (1',\lambda_3)=(2',\lambda_3)=w=0.$$

$$(2',\lambda_3)\equiv\begin{vmatrix}1&1&0&A\\ \lambda_1&\lambda_1^2&s(\lambda_1-\lambda_3)&B\\ \lambda_2&\lambda_2^2&t(\lambda_2-\lambda_3)&C\\ \lambda_3&\lambda_3^2&0&D\end{vmatrix}=0;\ (\sigma,2',\lambda_3)\ :\ (1',\lambda_3)=(2',\lambda_3)=\left.\frac{S-pT}{\lambda-\lambda_3}\right]_{\lambda\doteq\lambda_3}=0.$$

The symbol for this congruence is $\mathrm{II}(3,3)^{1,1}_{12,12}$.

31. As in the case of Type I, additional basis points and basis planes may be introduced and a configuration of σ and τ lines similar to that of Type I exists. These basis points, together with the necessary and sufficient conditions for their existence, are:

$$\begin{aligned} O_1&:\ (1',0)=(1',1)=(2',\infty)=0,\\ O_2&:\ (1',\infty)=(1',\lambda_1)=(2',0)=0,\\ O_3&:\ (1',\lambda_2)=(2',0)=(2',\infty)=0,\\ O_4&:\ a'=b'=w=0. \end{aligned}$$

For O_1:

$$\begin{vmatrix} 1 & 1 & 0 & 0 \\ \lambda_1 & \lambda_1^2 & s\lambda_1 & B' \\ \lambda_2 & \lambda_2^2 & t\lambda_2 & C' \\ \lambda_3 & \lambda_3^2 & p\lambda_3 & D' \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ \lambda_1 & s & s\lambda_1 & B' \\ \lambda_2 & t & t\lambda_2 & C' \\ \lambda_3 & p & p\lambda_3 & D' \end{vmatrix} = 0.$$

For O_2:

$$\begin{vmatrix} 1 & 0 & 0 & \bar{A} \\ \lambda_1^2 & s & s\lambda_1 & 0 \\ \lambda_2^2 & t & t\lambda_2 & \bar{C} \\ \lambda_3^2 & p & p\lambda_3 & \bar{D} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 & \bar{A} \\ \lambda_1 & \lambda_1^2 & s & 0 \\ \lambda_2 & \lambda_2^2 & t & \bar{C} \\ \lambda_3 & \lambda_3^2 & p & \bar{D} \end{vmatrix} = 0.$$

For O_3:

$$\begin{vmatrix} 1 & 1 & 0 & \bar{\bar{A}} \\ \lambda_1 & \lambda_1^2 & s & \bar{\bar{B}} \\ \lambda_2 & \lambda_2^2 & t & 0 \\ \lambda_3 & \lambda_3^2 & p & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & \bar{\bar{A}} \\ \lambda_1^2 & s & s\lambda_1 & \bar{\bar{B}} \\ \lambda_2^2 & t & t\lambda_2 & 0 \\ \lambda_3^2 & p & p\lambda_3 & 0 \end{vmatrix} = 0.$$

32. *It is possible to introduce into any of the preceding congruences one, two, three, four, five or six basis points without reducing the family of quadrics to a net of quadrics.*

Consider the family

$$H(\lambda)\equiv H_0\lambda^3+H_1\lambda^2+H_2\lambda+H_3=0.$$

$H_0=0$, $H_1=0$, $H_2=0$ have eight points in common. By imposing the condition that $H_3=0$ passes through this point, a basis point may be introduced into the family. This operation may be repeated for six distinct points, but if an attempt is made to introduce a seventh basis point, the eighth enters simultaneously, since the eight points commom to the three quadrics are associated points on a quartic curve.

33. It is to be noted that the points so chosen are not entirely arbitrary. If the points chosen are arbitrary points, it is not always possible to introduce the complete number. For example, in the cases of $\mathrm{I}(3,3)^{4,4}_{12,12}$, $\mathrm{II}(3,3)^{4,4}_{12,12}$,

discussed in Arts. 22, 31, it is impossible to introduce even a fifth basis point without the entire eight. An interesting example of this is the case of the congruences (4) and (5) when the vertices of the tetrahedron of reference are chosen as basis points. If these are basis points, the equation (4) takes the form:

$$\frac{(a_1x+a_2y)+(b_1x+b_2y)\lambda+(c_1x+c_2y)\lambda^2}{x+y\lambda}$$
$$=\frac{(a_3'z+a_4'w)+(b_3'z+b_4'w)\lambda+(c_3'z+c_4'w)\lambda^2}{z+w\lambda},$$

where $a\equiv a_1x+a_2y+a_3z+a_4w$, etc.

Let any other point x_1, y_1, z_1, w_1 be supposed to be a basis point; then any point of $x_1y=y_1x$, $z_1w=w_1z$ will also satisfy the identity. Hence no other point can be introduced as a basis point without introducing an entire line. The configuration of cones of σ and τ lines at the basis point is a K_6 of σ lines* and a K_3 of τ lines at each of the four basis points as a vertex. If, however, care is taken in the choice of points, it is often possible to introduce several rational points as basis points. An example of this is the following special case of

$$\frac{a+b\lambda+c\lambda^2}{x+y\lambda}=\frac{a'+b'\lambda+c'\lambda^2}{z+w\lambda}=\sigma \tag{4}$$

when (0, 0, 0, 1), (1, 0, 1, 0), (1, 1, 1, 1), (0, 1, 0, 1), (1, 0, 0, 0) are chosen as basis points. These five require the twelve conditions, which leave the congruence still (3, 9):

$$a_4=b_4=c_4=a_1+a_3-a_3'=b_1+b_3-b_3'=c_1+c_3-c_3'$$
$$=a_2-a_2'-a_4'=b_2-b_2'-b_4'=c_2-c_2'-c_4'=a_1'=b_1'=c_1'=0.$$

The resulting congruence is:

$$\frac{a_1x+a_2y+a_3z+(b_1x+b_2y+b_3z)\lambda+(c_1x+c_2y+c_3z)\lambda^2}{a_2'y+(a_1+a_3)z+(a_2-a_2')w+[b_2'y+(b_1+b_3)z+(b_2-b_2')w]\lambda+[c_2y+(c_1+c_3)z+(c_2-c_2')w]\lambda^2}=\frac{x+y\lambda}{z+w\lambda}=\tau.$$

The configurations of lines at the basis points are as follows: at (0, 0, 0, 1) and (1, 0, 0, 0), K_6^σ, and K_3^τ; at the remaining points, K_4^σ, K_5^τ.

For a set of six basis points (0, 0, 0, 1), (1, 0, 1, 0), (1, 1, 1, 1), (0, 1, 0, 1), (1, 0, 0, 0), (0, 1, 1, 0), the following sixteen conditions are necessary and sufficient if imposed on equations (4):

$a_1'=b_1'=c_1'=a_4=b_4=c_4=0$, $a_3=-a_2$, $a_2'=b_2+b_3-a_1+a_2$, $b_2'=b_2+b_3+c_2+c_3$, $c_2'=c_1-c_3$, $a_3'=a_1-a_2$, $b_3'=b_1+b_3$, $c_3'=c_1+c_3$, $a_4'=a_1-b_2-b_3$, $b_4'=-b_3-c_2-c_3$, $c_4'=c_1+c_2+c_3$.

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* For definition of symbols, see Snyder: *Transactions, loc. cit.*, p. 380.

The equations for the congruences are then:

$$\frac{N}{D}=\frac{x+y\lambda}{z+w\lambda}=\tau,$$

where

$$N\equiv a_1x+a_2(y-z)+(b_1x+b_2y+b_3z)\lambda+(c_1x+c_2y+c_3z)\lambda^2.$$

$$\begin{aligned}D\equiv{}&(b_2+b_3-a_1+a_2)y+(a_1-a_2)z+(a_1-b_2-b_3)w\\&+[(b_2+b_3+c_2+c_3)y+(b_1+b_3)z-(c_1+c_3+b_3)w]\lambda\\&+[-(c_1+c_3)y+(c_1+c_3)z+(c_1+c_2+c_3)w]\lambda^2.\end{aligned}$$

The configurations at all of the basis points are K_4^σ, K_5^τ, except at $(0,0,0,1)$ and $(1,0,0,0)$, where they are K_6^σ, K_3^τ. The introduction of any other basis point brings with it the eighth point, and the family of quadrics is contained in a net of quadrics through the basis points.

Congruences on Nets of Quadrics.

32. In order to determine the number of σ lines which coincide with τ lines of the conjugate congruences

$$\frac{a+b\lambda+c\lambda^2}{x+y\lambda}=\frac{a'+b'\lambda+c'\lambda^2}{z+w\lambda}=\sigma, \tag{4}$$

$$\frac{a+b\lambda+c\lambda^2}{a'+b'\lambda+c'\lambda^2}=\frac{x+y\lambda}{z+w\lambda}=\tau, \tag{5}$$

assume that the σ line for λ_1 coincides with the τ line for λ_2. Then

$$\begin{aligned}a+b\lambda_1+c\lambda_1^2+k(a'+b'\lambda_1+c'\lambda_1^2)-\sigma[x+y\lambda_1+k(z+w\lambda_1)]&\\ \equiv m[a+b\lambda_2+c\lambda_2^2-\tau(a'+b'\lambda_2+c'\lambda_2^2)]&;\end{aligned}$$

and

$$\begin{aligned}a+b\lambda_1+c\lambda_1^2+l(a'+b'\lambda_2+c'\lambda_2^2)-\sigma[x+y\lambda_1+l(z+w\lambda_1)]&\\ \equiv n[x+y\lambda_2-\tau(w+z\lambda_2)]&.\end{aligned}$$

Each of these identities requires three independent conditions; *i.e.*, a total of six conditions on the six constants λ_1, λ_2, σ, τ, k, l.

Hence only a finite number of σ lines can coincide with τ lines in the general (3, 9) *congruences without basis points.*

33. Consider the family of quadrics:

$$H(\lambda)\equiv\lambda^3Q_0+\lambda^2Q_1+\lambda Q_2+Q_3=0.$$

Let the powers of λ be considered as separate parameters, μ_1, μ_2, μ_3, μ_4. If these be considered as point coordinates in space, there is, corresponding to the family of quadrics, a twisted cubic. Taking the discriminant of this cubic, there results a quartic surface which intersects the cubic in twelve points.

From former considerations (Art. 6), these points must occur in coincident pairs; hence the surface and curve have six points of contact. If, then, it were possible for a λ_1 to exist such that all σ lines for that value were equal, respectively, to all τ lines for another value λ_2, *i.e.*, for a quadric surface of σ lines to coincide with a quadric surface of τ lines, or two quadrics of H to be coincident, the corresponding cubic must have a double point. This can happen only if the cubic degenerates.

Hence there are no surfaces of σ lines which coincide with surfaces of τ lines, excepting the six quadric cones of the system when the quadrics Q_i can not be expressed as linear functions of three of them.

34. If the quadrics Q_i of the family

$$H(\lambda) \equiv \lambda^3 Q_0 + \lambda^2 Q_1 + \lambda Q_2 + Q_3 = 0$$

can be expressed linearly in terms of three of them, the family must have eight basis points at the eight points of intersection of $Q_0=0$, $Q_1=0$, $Q_2=0$.

The family of quadrics

$$H(\lambda) \equiv \frac{a+b\lambda+c\lambda^2}{x+y\lambda} = \frac{a'+b'\lambda+c'\lambda^2}{z+w\lambda}$$

contains no basis point or basis plane unless restrictions are made on the constants. If this family be rewritten

$$\begin{aligned} H(\lambda) \equiv \lambda^3(cw-c'y) &+ \lambda^2(cz+bw-b'y-c'x) \\ &+ \lambda(aw+bz-a'y-b'x) + (az-a'x) = 0, \end{aligned}$$

and the powers of λ be considered as parameters μ_i, $i=1, \ldots, 4$, the relations which are equivalent to a linear relation among the fundamental quadrics of the family are:

$$\mu_1(cw-c'y) + \mu_2(bw+cz-b'y-c'x) + \mu_3(aw+bz-a'y-b'x) + (az-a'x) \equiv 0.$$

The necessary and sufficient condition for this identity is the consistency of the following ten equations:

$$\begin{aligned}
\mu_1 a'_1 + \mu_2 b'_1 + \mu_3 c'_1 &= 0, \\
\mu_2 a'_2 + \mu_3 b'_2 + \mu_4 c'_2 &= 0, \\
\mu_1 a_3 + \mu_2 b_3 + \mu_3 c_3 &= 0, \\
\mu_2 a_4 + \mu_3 b_4 + \mu_4 c_4 &= 0, \\
\mu_1 a'_2 + \mu_2(a'_1 + b'_2) + \mu_3(b'_1 + c'_2) + \mu_4 c'_1 &= 0, \\
\mu_1(a'_3 - a_1) + \mu_2(b_1 - b'_3) + \mu_3(c_1 - c'_3) &= 0, \\
\mu_1 a_2 + \mu_2(b_2 - a'_3) + \mu_3(c_2 - b'_3) - \mu_4 c'_3 &= 0, \\
\mu_1 a'_4 + \mu_2(b'_4 - a_1) + \mu_3(c'_4 - b_1) - \mu_4 c_1 &= 0, \\
\mu_2(a_2 - a'_4) + \mu_3(b_2 - b'_4) + \mu_4(c_2 - c'_4) &= 0, \\
\mu_1 a_4 + \mu_2(a_3 + b_4) + \mu_3(b_3 + c_4) + \mu_4 c_3 &= 0.
\end{aligned}$$

For this the necessary and sufficient condition is:

$$\left\|\begin{matrix} a_1' & 0 & a_3 & 0 & a_2' & (a_3'-a_1) & a_2 & a_4' & 0 & a_4 \\ b_1' & a_2' & b_3 & a_4 & (a_1'+b_2') & (b_1-b_3') & (b_2-a_3') & (b_4'-a_1) & (a_2-a_4') & (a_3+b_4) \\ c_1' & b_2' & c_3 & b_4 & (b_1'+c_2') & (c_1-c_3') & (c_2-b_3') & (c_4'-b_1) & (b_2-b_4') & (c_4+b_3) \\ 0 & c_2' & 0 & c_4 & (c_1') & (0) & -c_3' & -c_1 & (c_2-c_4') & c_3 \end{matrix}\right\| = 0.$$

The matrix is equal to zero if the following seven determinants are zero:

$$\begin{vmatrix} c_3 & c_1' & -c_3' & m_i \\ b_3+c_4 & b_1'+c_2' & c_2-b_3' & n_i \\ a_3+b_4 & a_1'+b_2' & b_2-a_3' & p_i \\ a_4 & a_2' & a_2 & q_i \end{vmatrix} = 0, \quad i=1,\ldots\ldots,7,$$

the fourth column being replaced in turn by the seven columns of the matrix remaining after picking out the above three.

Hence, by imposing seven independent conditions upon the constants of the congruences (4), (5), *it is possible to introduce eight basis points simultaneously.*

35. Suppose there exist two values of λ (λ_1, λ_2) such that for these values the corresponding quadrics of H are identical. The condition for the possibility of this is the consistency of ten simultaneous equations in the four variables $(\lambda_1^3-k\lambda_2^3)$, $(\lambda_1^2-k\lambda_2^2)$, $(\lambda_1-k\lambda_2)$, $(1-k)$. If these variables are called μ_i, these ten equations are identical with those of the preceding paragraph, the consistency of which is the necessary and sufficient condition for the introduction of a linear relation among the fundamental quadrics.

Every rational pair of conjugate congruences which has its lines arranged on a net of quadrics contains all the lines of one quadric as double lines of the congruence. This quadric will be called the double quadric of the congruence.

It will be shown later that in the congruence $\mathrm{I}(3,9)^8$ every line of the double quadric is either twice a σ line or twice a τ line, *i.e.*, one system of generators is composed of double lines of the σ congruence and the other system is composed of double lines of the τ congruence

36. The quadrics of the family

$$H(\lambda) \equiv \frac{a+b\lambda+c\lambda^2}{x+y\lambda+z\lambda^2} = \frac{a'+b'\lambda}{w} = \sigma$$

contain a basis point $a'=b'=w=0$, and a basis plane, $w=0$. Introducing seven additional basis points in this family introduces simultaneously a double quadric, since in each case the necessary and sufficient condition is:

$$\left\|\begin{matrix} a'_1 & 0 & 0 & a_4 & a'_2 & a'_3 & a_1-a'_4 & 0 & a_2 & a_3 \\ b'_1 & a'_2 & 0 & b_4 & a'_1+b'_2 & b'_3 & b_1-b'_4 & a'_3 & b_2-a'_4 & b_3 \\ 0 & b'_2 & a'_3 & c_4 & b'_1 & a'_1 & c_1 & a_2+b'_3 & c_2-b'_4 & c_3-a'_4 \\ 0 & 0 & b'_3 & d_4 & 0 & b'_1 & d_1 & b'_2 & d_2 & d_3-b'_4 \end{matrix}\right\|=0.$$

The conditions for this are the vanishing of the seven determinants

$$\begin{vmatrix} a'_1 & 0 & 0 & m_i \\ b'_1 & a'_2 & 0 & n_i \\ 0 & b'_2 & a'_3 & p_i \\ 0 & 0 & b'_3 & q_i \end{vmatrix}=0, \quad i=1, \ldots, 7,$$

the last column being replaced in turn by the seven columns of the matrix remaining after picking out the above three. These seven conditions reduce to six independent ones, since the two determinants

$$\begin{vmatrix} a'_1 & 0 & 0 & 0 \\ b'_1 & a'_2 & 0 & a'_3 \\ 0 & b'_2 & a'_3 & a'_2+b'_3 \\ 0 & 0 & b'_3 & b'_2 \end{vmatrix} = a'_1(a'_2+b'_3)(b'_2a'_3-a'_2b'_3)=0;$$

hence $b'_2a'_3=a'_2b'_3$, since $a'_2+b'_3\neq0$, and $a'_1\neq0$; and

$$\begin{vmatrix} a'_1 & 0 & 0 & a'_2 \\ b'_1 & a'_2 & 0 & a'_1+b'_2 \\ 0 & b'_2 & a'_3 & b'_1 \\ 0 & 0 & b'_3 & 0 \end{vmatrix} = 0=b'_3(a'_1+b'_2)(a'_2b'_1-a'_1b'_2);$$

hence $a'_1b'_2-a'_2b'_1=0$, since $b'_3\neq0$, and $a'_1+b'_2\neq0$, are equivalent to the single condition $a'_1b'_3-a'_3b'_1=0$. Hence this insures the vanishing of the determinant:

$$\begin{vmatrix} a'_1 & 0 & 0 & a'_3 \\ b'_1 & a'_2 & 0 & b'_3 \\ 0 & b'_2 & a'_3 & a'_1 \\ 0 & 0 & b'_3 & b'_1 \end{vmatrix} = (a'_1a'_2-b'_2b'_3)(a'_1b'_3-a'_3b'_1).$$

Hence, for the congruences (6) *and* (7), *six independent conditions are necessary and sufficient to introduce seven additional basis points and a double quadric.*

37. In the case of all congruences of Type II, the plane $w=0$ is always a basis plane, and therefore is tangent to each quadric of the family. It will contain one generator of each system on the double quadric. Through this point the two τ lines are $w=a'+b'\lambda_1=0$ and $w=a'+b'\lambda_2=0$. Hence, on the double quadric the basis plane contains a τ line as a generator of each system once. This is equivalent to saying that the τ lines of the quadric $Q(\lambda_1)$ form

one system, and the τ lines of the quadric $Q(\lambda_2)$ form the second system of generators of the double quadric.

Each line of the double quadric for the conjugate congruences $\mathrm{II}(3,9)^{8,1}$ *is once a τ line and once a σ line.*

The only condition under which a congruence of Type II can contain a double quadric of the $2\sigma, 2\tau$ type, is that one of the six quadric cones should be a double quadric. This can happen only if the congruence is of class less than 9, hence if the cone degenerates to a pair of planes.

38. In any congruence it is possible to determine the relation between the class of the congruence and the order of the singular cone whose vertex is at a basis point.

For any family of quadrics

$$Q_0\lambda^n+Q_1\lambda^{n-1}+Q_2\lambda^{n-2}+\ldots\ldots=0,$$

the tangent plane has the equation

$$V_0\lambda^n+V_1\lambda^{n-1}+V_2\lambda^{n-2}+\ldots\ldots=0,$$

where Q_i is of order 2, V_i is of order 1. The λ eliminant of these two equations is the equation of the singular cone at a basis point. In the general case this will be K_{3n}; but if any cone of the family is composed of planes, there will appear a factor in the equation and the eliminant is K_{3n-i}, where i is the number of cones of the family which are composed of planes. But the class of the congruence is $3n-i$.

The class of a congruence is equal to the order of the singular cone whose vertex is at a basis point; i. e., is equal to the sum of the orders of the σ and τ configurations at any basis point.

39. Every line joining a pair of basis points is, in general, a triple line of the composite singular cone belonging to a pair of conjugate congruences. If these two points lie in a singular plane, the multiplicity of the line is decreased by one. Hence, if three singular planes pass through a pair of basis points, the line joining them becomes a neutral line. In all the work that follows, the notation $K_n(i; j_r)$ will be used to indicate a cone of order n of σ lines, vertex at basis point i; having the line ij as an r-fold line. The same notation with K' indicates a cone of τ lines.

40. If the coordinates of a basis point be substituted in the equations which define a congruence, these equations become indeterminate. This may happen in any one of three ways: (1) in one of the fractions both numerator and denominator may vanish simultaneously; (2) both numerators may vanish;

or (3) both denominators may vanish. The third of these possibilities has been eliminated in congruences (4), ..., (7) by the choice of coordinate axes.

It is possible, from consideration of the equations and the method of procedure due to Snyder,* to write in full the σ and τ configurations at each basis point and to determine the congruences which may be written in the form of Type I or Type II.

41. For the case of a (3,9) congruence there are four pairs of conjugate congruences distinguished by their configurations. The first of these is that determined by $K_1(1)$, a basis plane through one and only one basis point. When conditions for eight basis points are imposed on $\text{II}(3,9)^{1,1}$, the point $a'=b'=w$ is still a basis point. Then the configuration of τ lines was shown in Art. 23 to be a pencil and the configuration of σ lines a cone of order 8. No other basis point lies in the plane $w=0$. Hence all the other basis points must lie on the K_8. Further, every line joining two basis points is a triple line on K_8. The complete configuration for this congruence is:

$$\begin{cases} K_8(1;\, 2_3\, 3_3\, 4_3\, 5_3\, 6_3\, 7_3\, 8_3) \\ K'_1(1) \end{cases} \qquad \begin{cases} K_4(2;\, 1_3\, 3_1\, 4_1\, 5_1\, 6_1\, 7_1\, 8_1) \\ K'_5(2;\, 3_2\, 4_2\, 5_2\, 6_2\, 7_2\, 8_2) \end{cases}$$

Cones similar to those at (2) exist at points 3, ..., 8. The double quadric in this case is composed of lines each of which is once a σ line and once a τ line.

42. For the case of $\text{I}(3,9)^8_6$ the basis points may be classified according as their coordinates satisfy

$$\text{(A)}\quad a+b\lambda+c\lambda^2-k(a'+b'\lambda+c'\lambda^2)=0,\quad x+y\lambda-k(z+w\lambda)=0;$$
$$\text{(B)}\quad a'+b'\lambda+c'\lambda^2=0,\quad a+b\lambda+c\lambda^2=0.$$

If condition (A) is fulfilled,

$$\frac{\bar{a}+\bar{b}\lambda+\bar{c}\lambda^2}{\bar{x}+\bar{y}\lambda}=\frac{F_2(\lambda)}{f_1(\lambda)}=\bar{\sigma},$$

where the dash over the letter indicates its particular value when the coordinates of the basis point have been substituted. The equations for all σ lines through this point are:

$$\frac{a+b\lambda+c\lambda^2}{x+y\lambda}=\frac{a'+b'\lambda+c'\lambda^2}{z+w\lambda}=\frac{F_2(\lambda)}{f_1(\lambda)}.$$

For the configuration of all σ lines, therefore, there exists a cone of order 6. For the τ lines the configuration is the eliminant of the equations

$$a+b\lambda+c\lambda^2=x+y\lambda=0.$$

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* Snyder: *Transactions, loc. cit.*

This is a cubic cone, as was to be expected from the preceding work and Art. 38. This pair of cones fixes an entire configuration as soon as the lines of K_3^τ are fixed.* The cubic cone must have six lines joining basis points, and it must not pass through all eight basis points, since these eight points lie on a quartic curve and the cubic cone defined by this quartic with vertex on the curve would define a (2,6) congruence. Hence there are left but two possibilities: either the cone has simple lines through six basis points or it has simple lines through four points and a double line through the fifth.

The first case, in which the double quadric has each line either twice σ or twice τ, has the complete configuration:

$$\begin{array}{ll}
\begin{cases} K_6(1;\, 2_2 3_2 4_2 5_2 6_2 7_2 8_3) \\ K_3'(1;\, 2_1 3_1 4_1 5_1 6_1 7_1) \end{cases} &
\begin{cases} K_4(2;\, 1_2 3_1 4_1 5_1 6_1 7_1 8_2) \\ K_5'(2;\, 1_1 3_2 4_2 5_2 6_2 7_2 8_1) \end{cases} \\
\begin{cases} K_4(3;\, 1_2 2_1 4_1 5_1 6_1 7_1 8_2) \\ K_5'(3;\, 1_1 2_2 4_2 5_2 6_2 7_2 8_1) \end{cases} &
\begin{cases} K_4(4;\, 1_2 2_1 3_1 5_1 6_1 7_1 8_2) \\ K_5'(4;\, 1_1 2_2 3_2 5_2 6_2 7_2 8_1) \end{cases} \\
\begin{cases} K_4(5;\, 1_2 2_1 3_1 4_1 6_1 7_1 8_2) \\ K_5'(5;\, 1_1 2_2 3_2 4_2 6_2 7_2 8_1) \end{cases} &
\begin{cases} K_4(6;\, 1_2 2_1 3_1 4_1 5_1 7_1 8_2) \\ K_5'(6;\, 1_1 2_2 3_2 4_2 5_2 7_2 8_1) \end{cases} \\
\begin{cases} K_4(7;\, 1_2 2_1 3_1 4_1 5_1 6_1 8_2) \\ K_5'(7;\, 1_1 2_2 3_2 4_2 5_2 6_2 8_1) \end{cases} &
\begin{cases} K_6(8;\, 1_3 2_2 3_2 4_2 5_2 6_2 7_2) \\ K_3'(8;\, 2_1 3_1 4_1 5_1 6_1 7_1) \end{cases}
\end{array}$$

Since all the cones are rational, each of the cones has a double edge which is not an *ik* line. This double line must lie on the double quadric. If the basis point is of the type (B), the σ lines form a K_4. The only possibilities for this are $K_4(1;\, 2_2 3_2 4_1 5_1 6_1 7_1 8_1)$ or $K_4(1;\, 2_2 3_2 4_2 5_1 6_1 7_1)$. The second of these requires a K_2 which can not occur, and the first gives the same configuration as $K_3'(1;\, 2_1 3_1 4_1 5_1 6_1 7_1)$. Hence the above configuration is the only possible one for the general congruence $\mathrm{I}(3,9)_6^8$.

43. If, in the congruences (4) and (5), the denominators are taken as linear functions of the variables, the congruences become

$$\frac{a+b\lambda+c\lambda^2}{m+n\lambda}=\frac{a'+b'\lambda+c'\lambda^2}{m'+n'\lambda}=\sigma. \quad [\mathrm{III}(3,9).]$$

If the conditions for eight basis points be imposed on these equations, the possibilities for a configuration will include those of Art. 42 and, in addition to these, a $K_2'(1;\, 3_1 4_1 5_1)$. The complete configurations for these are:

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* Snyder: *Transactions, loc. cit.*

$$\left\{\begin{array}{l} K_7(1;\ 2_3\,3_2\,4_2\,5_2\,6_3\,7_3\,8_3) \\ K'_2(1;\ 3_1\,4_1\,5_1) \end{array}\right. \qquad \left\{\begin{array}{l} K_5(2;\ 1_3\,3_1\,4_1\,5_1\,6_2\,7_2\,8_2) \\ K'_4(2;\ 3_2\,4_2\,5_2\,6_1\,7_1\,8_1) \end{array}\right.$$

$$\left\{\begin{array}{l} K_3(3;\ 1_2\,2_1\,6_1\,7_1\,8_1) \\ K'_6(3;\ 1_1\,2_2\,4_3\,5_3\,6_2\,7_2\,8_2) \end{array}\right. \qquad \left\{\begin{array}{l} K_3(4;\ 1_2\,2_1\,6_1\,7_1\,8_1) \\ K'_6(4;\ 1_1\,2_2\,3_3\,5_3\,6_2\,7_2\,8_2) \end{array}\right.$$

$$\left\{\begin{array}{l} K_3(5;\ 1_2\,2_1\,6_1\,7_1\,8_1) \\ K'_6(5:\ 1_1\,2_2\,3_3\,4_3\,6_2\,7_2\,8_2) \end{array}\right. \qquad \left\{\begin{array}{l} K_5(6;\ 1_3\,2_2\,3_1\,4_1\,5_1\,7_2\,8_2) \\ K'_4(6;\ 2_1\,3_2\,4_2\,5_2\,7_1\,8_1) \end{array}\right.$$

$$\left\{\begin{array}{l} K_5(7;\ 1_3\,2_2\,3_1\,4_1\,5_1\,6_2\,8_2) \\ K'_4(7;\ 2_1\,3_2\,4_2\,5_2\,6_1\,8_1) \end{array}\right. \qquad \left\{\begin{array}{l} K_5(8;\ 1_3\,2_2\,3_1\,4_1\,5_1\,6_2\,7_2) \\ K'_4(8;\ 2_1\,3_2\,4_2\,5_2\,6_1\,7_1) \end{array}\right.$$

The double quadric is a σ, τ in this case. By imposing conditions on the basis points that they shall lie by fours in a pair of planes, etc., it is possible to obtain (3,9) congruences which have the basis points arranged by fours in one, two or three pairs of planes. It is impossible to have a (3,9) congruence with the basis points by fours in more than three pairs of planes.

44. If a (3,8) congruence lies on a net of quadrics, it follows from the previous discussion that the net will include a pair of planes each of which contains a pencil of σ lines and a pencil of τ lines. Hence the basis points will be arranged in two planes. Moreover, they lie four in each plane, since if five or more should lie in one plane, they must all lie on a conic which, being uniquely fixed, would be common to every quadric of the family, which is contrary to our hypothesis. In the case of I(3,8), the basis point satisfies one of the following conditions:

$$\begin{array}{ll} \text{(A)} & b\lambda+c\lambda^2=a'+b'\lambda+c'\lambda^2=0, \\ \text{(B)} & a'+b'\lambda+c'\lambda^2=x+y\lambda=0, \\ \text{(C)} & b\lambda+c\lambda^2=x+y\lambda=0. \end{array}$$

The only distinction between (A) and (B) is that the σ and τ configurations are interchanged throughout. This gives a configuration K_3, K'_5. For the basis point (C) there exists a K_6, K'_2. In the case of II(3,8), the basis point satisfies

(A) $a'=b'=w=0$, giving the configuration K_7, K'_1;
(B) $b\lambda+c\lambda^2+d\lambda^3=x+y\lambda+z\lambda^2=0$, giving K_4, K'_4;
(C) $b\lambda+c\lambda^2+d\lambda^3=a'+b'\lambda=0$, giving K_3, K'_5.

The case of III(3,8) does not contribute anything different from the congruences already discussed. The only possible K'_2 which gives a (3,8) congruence is $K'_2(1;\ 2_1\,5_1\,8_1)$. The double quadric in this case is of the σ, τ type. The complete configuration is:

$$\left\{\begin{array}{l} K_6(1;\, 2_1 3_2 4_2 5_2 6_3 7_3 8_2) \\ K_2'(1;\, 2_1 5_1 8_1) \end{array}\right. \qquad \left\{\begin{array}{l} K_2(2;\, 1_1 6_1 7_1) \\ K_6'(2;\, 1_1 3_2 4_2 5_3 6_2 7_2 8_3) \end{array}\right.$$

$$\left\{\begin{array}{l} K_4(3;\, 1_2 4_1 5_1 6_2 7_2 8_1) \\ K_4'(3;\, 2_2 4_1 5_2 6_1 7_1 8_2) \end{array}\right. \qquad \left\{\begin{array}{l} K_3(5;\, 1_2 3_1 4_1 6_1 7_1) \\ K_5'(5;\, 1_1 2_3 3_2 4_2 6_1 7_1 8_2) \end{array}\right.$$

$$\left\{\begin{array}{l} K_5(6;\, 1_3 2_1 3_2 4_2 5_1 7_2 8_1) \\ K_3'(6;\, 2_2 3_1 4_1 5_1 8_1) \end{array}\right.$$

The cones at (4), (8) and (7) are similar respectively to the ones at (3), (5) and (6).

There are two K_3 which give (3,8) congruences different from the above: $K_3(1;\, 2_1 3_1 5_1 6_1 7_1 8_1)$ and $K_3(1;\, 2_2 5_1 6_1 7_1 8_1)$. For the first of these the double quadric is of the 2σ, 2τ type. The complete configuration is:

$$\left\{\begin{array}{l} K_3(1;\, 2_1 3_1 5_1 6_1 7_1 8_1) \\ K_5'(1;\, 2_1 3_1 4_2 5_2 6_2 7_2 8_2) \end{array}\right. \qquad \left\{\begin{array}{l} K_5(2;\, 1_1 3_2 4_1 5_2 6_2 7_2 8_2) \\ K_3'(2;\, 1_1 4_1 5_1 6_1 7_1 8_1) \end{array}\right.$$

$$\left\{\begin{array}{l} K_4(5;\, 1_1 2_2 3_2 4_1 6_1 7_1 8_1) \\ K_4'(5;\, 1_2 2_1 3_1 4_2 6_1 7_1 8_1) \end{array}\right.$$

The cones for (3) and (4) are similar to those for (2) and (1), respectively, while those for (6), (7) and (8) are similar to (5). The complete configuration for the third case, in which the double quadric is of the σ, τ type, is the same as that determined by the K_1. It is as follows:

$$\left\{\begin{array}{l} K_1'(1) \\ K_7(1;\, 2_2 3_2 4_2 5_3 6_3 7_3 8_3) \end{array}\right. \qquad \left\{\begin{array}{l} K_4'(5;\, 2_2 3_2 4_2 6_1 7_1 8_1) \\ K_4(5;\, 1_3 2_2 3_2 4_2 6_1 7_1 8_1) \end{array}\right.$$

$$\left\{\begin{array}{l} K_5'(3;\, 2_2 4_2 5_2 6_2 7_2 8_2) \\ K_3(3;\, 1_2 5_1 6_1 7_1 8_1) \end{array}\right.$$

The cones for (2) and (4) are similar to that at (3); those for (6), (7), (8) are similar to that at (5).

By imposing conditions on the basis points it is possible to obtain a (3,8) congruence with the basis points arranged by fours on two, three, four or five pairs of planes. It is impossible to have a (3,8) congruence with the basis points arranged on six pairs of planes.

45. In a net of quadrics containing six cones and one double quadric, it is possible that one of the cones may be the double quadric, in case the cone is composed of planes. In this case, there are included in the net only four cones, and one pair of planes each of which is counted twice. In general the σ and τ vertices do not coincide.

If this occurs in the case just considered, it is possible to have a (3,7) congruence in which the basis points lie by fours in only one pair of planes; *i. e.*, in the planes of pencils of lines belonging to the congruence. The complete configuration is as follows:

$$\begin{cases} K_2'(1;\ 5_1\,6_1\,7_1) \\ K_5(1;\ 2_1\,3_1\,4_1\,5_2\,6_2\,7_2\,8_3) \end{cases} \qquad \begin{cases} K_4'(2;\ 3_1\,4_1\,5_2\,6_2\,7_2\,8_1) \\ K_3(2;\ 1_1\,5_1\,6_1\,7_1\,8_2) \end{cases}$$

$$\begin{cases} K_4'(5;\ 1_1\,2_2\,3_2\,4_2\,6_1\,7_1) \\ K_3(5;\ 1_2\,2_1\,3_1\,4_1\,8_1) \end{cases}$$

The cones for (1) and (8) are similar to (1); those for (3) and (4) are similar to (2); and those for (6) and (7) are similar to (5).

46. If a (3,7) congruence lies on a net of quadrics, it has been shown that the net must in general contain two pairs of planes. Hence the basis points lie by fours in these four planes. As in the preceding cases, the basis points in I(3,7) may be shown to give a K_2, K_5'; those in II(3,7) will give K_6, K_1' and K_4, K_3'. The III(3,7) will add nothing to these. K_1' determines a single pair of congruences with a double quadric of the σ, τ type. The configuration follows:

$$\begin{cases} K_6(1;\ 2_1\,3_2\,4_2\,5_3\,6_3\,7_2\,8_2) \\ K_1'(1) \end{cases} \qquad \begin{cases} K_2(2;\ 1_1\,5_1\,6_1) \\ K_5'(2;\ 3_2\,4_2\,5_2\,6_2\,7_2\,8_2) \end{cases}$$

$$\begin{cases} K_3(3;\ 1_2\,5_1\,6_1\,7_1\,8_1) \\ K_4'(3;\ 2_2\,4_1\,5_1\,6_1\,7_2\,8_2) \end{cases} \qquad \begin{cases} K_4(5;\ 1_3\,2_1\,3_1\,4_1\,6_1\,7_1\,8_1) \\ K_3'(5;\ 2_2\,3_1\,4_1\,7_1\,8_1) \end{cases}$$

The cones for (4), (7) and (8) are similar to (3), and that for (6) is similar to (5).

There are two distinct K_2 which give (3,7) congruences. The first of these, $K_2(1;\ 2_1\,5_1\,6_1)$, is identical with the congruence determined by K_1'. For the second, $K_2(1;\ 3_1\,5_1\,7_1)$, the congruence has the double quadric of the σ, τ type, and the configuration is:

$$\begin{cases} K_2(1;\ 3_1\,5_1\,7_1) \\ K_5'(1;\ 2_1\,3_1\,4_2\,5_2\,6_3\,7_1\,8_2) \end{cases} \qquad \begin{cases} K_4(2;\ 3_2\,4_1\,5_2\,6_1\,7_2\,8_1) \\ K_3'(2;\ 1_1\,4_1\,5_1\,6_2\,8_1) \end{cases}$$

$$\begin{cases} K_5(3;\ 1_1\,2_2\,4_1\,5_2\,6_1\,7_3\,8_2) \\ K_2'(3;\ 1_1\,6_1\,8_1) \end{cases} \qquad \begin{cases} K_3(4;\ 2_1\,3_1\,5_1\,7_2\,8_1) \\ K_4'(4;\ 1_2\,2_1\,5_1\,6_2\,7_1\,8_2) \end{cases}$$

$$\begin{cases} K_2(6;\ 2_1\,3_1\,7_1) \\ K_5'(6;\ 1_3\,2_2\,3_1\,4_2\,5_1\,7_1\,8_2) \end{cases}$$

The cones at (5) and (7) are similar to (3), and (8) is similar to (6).

Of the K_3, the only one which gives a new congruence is

$$K_3(1;\ 3_1\,4_1\,5_1\,6_1\,7_1\,8_1).$$

The double quadric for this is of the $2\sigma, 2\tau$ type. The configuration is:

$$\left\{\begin{array}{l}K_3(1;\ 3_1\,4_1\,5_1\,6_1\,7_1\,8_1)\\ K'_4(1;\ 2_1\,3_1\,4_1\,5_2\,6_2\,7_1\,8_1)\end{array}\right. \qquad \left\{\begin{array}{l}K_3(2;\ 3_1\,4_1\,5_1\,6_1\,7_1\,8_1)\\ K'_4(2;\ 1_1\,3_1\,4_1\,5_2\,6_2\,7_1\,8_1)\end{array}\right.$$

$$\left\{\begin{array}{l}K_3(3;\ 1_1\,2_1\,5_1\,6_1\,7_1\,8_1)\\ K'_4(3;\ 1_1\,2_1\,4_1\,5_1\,6_1\,7_2\,8_2)\end{array}\right. \qquad \left\{\begin{array}{l}K_3(5;\ 1_1\,2_1\,3_1\,4_1\,7_1\,8_1)\\ K'_4(5;\ 1_2\,2_2\,3_1\,4_1\,6_1\,7_1\,8_1)\end{array}\right.$$

The cones at (4), (7), (8) are similar to that at (3), and that at (6) is similar to that at (5).

By imposing conditions on the basis points, (3, 7) congruences may be built having their basis points by fours in six, eight, ten or twelve planes.

47. Just as in Art. 45 it was possible to obtain a (3, 7) with the basis points in only one pair of planes, so, if one of the pairs of singular planes (1278), (3456) of Art. 46 is the double quadric of the net, the configuration $K_2(1;\ 3_1\,5_1\,6_1)$ determines a (3, 6) congruence, with the pair of planes as the double quadric and the basis points arranged by fours only in two pairs of planes. The configuration follows:

$$\left\{\begin{array}{l}K_2(1;\ 3_1\,5_1\,6_1)\\ K'_4(1;\ 3_1\,4_2\,5_2\,6_2\,7_1\,8_1)\end{array}\right. \qquad \left\{\begin{array}{l}K_4(2;\ 3_2\,4_1\,5_2\,6_2\,7_1\,8_1)\\ K'_2(2;\ 4_1\,5_1\,6_1)\end{array}\right.$$

$$\left\{\begin{array}{l}K_4(3;\ 1_1\,2_2\,5_1\,6_1\,7_2\,8_2)\\ K'_2(3;\ 1_1\,7_1\,8_1)\end{array}\right. \qquad \left\{\begin{array}{l}K_2(4;\ 2_1\,7_1\,8_1)\\ K'_4(4;\ 1_2\,2_1\,5_1\,6_1\,7_2\,8_2)\end{array}\right.$$

$$\left\{\begin{array}{l}K_3(5;\ 1_1\,2_2\,3_1\,7_1\,8_1)\\ K'_3(5;\ 1_2\,2_1\,4_1\,7_1\,8_1)\end{array}\right. \qquad \left\{\begin{array}{l}K_3(7;\ 2_1\,3_2\,4_1\,5_1\,6_1)\\ K'_3(7;\ 3_1\,4_2\,5_1\,6_1\,8_1)\end{array}\right.$$

The cone at (6) is similar to that at (5), and that at (8) is similar to that at (7).

48. In the case of three pairs of singular planes, *i. e.*, the (3, 6) congruences, there arise two distinct classes according as the three pairs of planes have general position or the planes pass by threes through a pair of lines. In either case, from II(3, 6), there arise K_5, K'_1 and K_2, K'_4. I(3, 6) adds nothing new to this, but III(3, 6) gives a K_3, K'_3.

For cases of the first class, the double quadrics are all of the σ, τ type. The complete configurations follow:

$$\left\{\begin{array}{l}K'_1(1)\\ K_5(1;\ 2_1\,3_2\,4_1\,5_3\,6_2\,7_1\,8_2)\end{array}\right. \qquad \left\{\begin{array}{l}K'_4(2;\ 3_1\,4_2\,5_1\,6_2\,7_2)\\ K_2(2;\ 1_1\,5_1\,8_1)\end{array}\right.$$

$$\left\{\begin{array}{l}K'_3(3;\ 2_1\,4_1\,6_1\,8_1\,7_2)\\ K_3(3;\ 1_2\,5_1\,6_1\,7_1\,8_1)\end{array}\right. \qquad \left\{\begin{array}{l}K'_2(5;\ 2_1\,4_1\,7_1)\\ K_4(5;\ 1_3\,2_1\,3_1\,4_1\,6_1\,7_1\,8_1)\end{array}\right.$$

The cone for (4) is similar to that at (2); those for (6), (7) and (8) are similar to that at (3).

$$\left\{\begin{array}{l} K_2'(1;\, 2_1\,3_1\,5_1) \\ K_4(1;\, 3_1\,4_1\,5_2\,6_2\,7_1\,8_2) \end{array}\right. \qquad \left\{\begin{array}{l} K_4'(2;\, 1_1\,3_1\,4_2\,5_1\,6_2\,7_2) \\ K_2(2;\, 5_1\,6_1\,8_1) \end{array}\right.$$

$$\left\{\begin{array}{l} K_3'(3;\, 1_1\,2_1\,4_1\,6_1\,7_2) \\ K_3(3;\, 1_1\,5_1\,6_1\,7_1\,8_2) \end{array}\right. \qquad \left\{\begin{array}{l} K_2'(8;\, 4_1\,6_1\,7_1) \\ K_4(8;\, 1_2\,2_1\,3_2\,4_2\,5_1\,6_1) \end{array}\right.$$

The cones for (4), (5) and (6) are similar to that at (3); the cone for (7) is similar to that at (2).

In the second class, there are three distinct congruences. For each of the first two, the double quadric is of the σ, τ type. The configurations are:

$$\left\{\begin{array}{l} K_1'(1) \\ K_5(1;\, 3_2\,4_2\,5_2\,6_2\,7_2\,8_2) \end{array}\right. \qquad \left\{\begin{array}{l} K_5'(2;\, 3_2\,4_2\,5_2\,6_2\,7_2\,8_2) \\ K_1(2) \end{array}\right.$$

$$\left\{\begin{array}{l} K_3'(3;\, 2_2\,5_1\,6_1\,7_1\,8_1) \\ K_3(3;\, 1_2\,5_1\,6_1\,7_1\,8_1) \end{array}\right.$$

The cones at (4), (5), (6), (7), (8) are similar to those at (3).

$$\left\{\begin{array}{l} K_2(1;\, 3_1\,4_1\,5_1) \\ K_4'(1;\, 3_1\,4_1\,5_1\,6_2\,7_2\,8_2) \end{array}\right. \qquad \left\{\begin{array}{l} K_4(2;\, 3_2\,4_2\,5_2\,6_1\,7_1\,8_1) \\ K_2'(2;\, 6_1\,7_1\,8_1) \end{array}\right.$$

$$\left\{\begin{array}{l} K_3(3;\, 1_1\,2_2\,5_1\,7_1\,8_1) \\ K_3'(3;\, 1_1\,6_2\,5_1\,7_1\,8_1) \end{array}\right. \qquad \left\{\begin{array}{l} K_2(6;\, 2_1\,7_1\,8_1) \\ K_4'(6;\, 1_2\,2_1\,3_2\,4_2\,7_1\,8_1) \end{array}\right.$$

$$\left\{\begin{array}{l} K_3(7;\, 2_1\,3_1\,4_1\,5_2\,6_1) \\ K_3'(7;\, 1_2\,2_1\,3_1\,4_1\,6_1) \end{array}\right.$$

The cones at (4) and (5) are similar to (3), and (8) is similar to (7).

Probably the most interesting of all is the (3, 6) defined by K_3, K_3' with a $2\sigma, 2\tau$ double quadric. In this case the σ lines and τ lines enter in perfect symmetry, and this is one of the two congruences of the list in which all the cones are identical in structure. The complete configuration is:

$$\left\{\begin{array}{l} K_3(1;\, 3_1\,4_1\,5_1\,6_1\,7_1\,8_1) \\ K_3'(1;\, 3_1\,4_1\,5_1\,6_1\,7_1\,8_1) \end{array}\right.$$

and similar configurations for all the other vertices. It is to be noted that the basis point omitted in each cone is the one such that the line joining it to the vertex of the cone is a neutral line, *i. e.*, lies in three singular planes. By imposing conditions, it is possible to put the basis points on eight, ten or twelve planes.

49. In the second class of congruences discussed in Art. 48, it is impossible that any pair of singular planes should be a double quadric, as that would make one line lie in four singular planes, which can not be true with congruences of order 3. In the first class, however, one of the pairs of planes (1467), (2358) may be the double quadric, in which case there arises a (3,5) congruence with the basis points by fours in each of six planes. These are the configurations:

$$\left\{\begin{array}{l} K_2(1;\, 3_1 5_1 8_1) \\ K_3'(1;\, 2_1 3_1 5_2 6_1) \end{array}\right. \qquad \left\{\begin{array}{l} K_3(3;\, 1_1 4_1 6_1 7_2 8_1) \\ K_2'(3;\, 1_1 6_1 7_1) \end{array}\right.$$

The cones at (2), (5) and (6) are similar to (1), and (4), (7) and (8) are similar to (3).

50. Proceeding as before, there arise from I(3,5) a K_2, K_3' and from II(3,5) an additional K_4, K_1' which gives the same (3,5) congruence, with the basis points lying by fours in eight singular planes. It is impossible to have a $2\sigma, 2\tau$ double quadric for a congruence of class lower than 6. The configurations for this case are:

$$\left\{\begin{array}{l} K_4(1;\, 3_2 4_1 5_2 6_1 7_1 8_2) \\ K_1'(1) \end{array}\right. \qquad \left\{\begin{array}{l} K_1(2) \\ K_4'(2;\, 3_1 4_2 5_1 6_2 7_2 8_1) \end{array}\right.$$

$$\left\{\begin{array}{l} K_3(3;\, 1_2 5_1 6_1 7_1 8_1) \\ K_2'(3;\, 2_1 6_1 7_1) \end{array}\right. \qquad \left\{\begin{array}{l} K_2(4;\, 1_1 5_1 8_1) \\ K_3'(4;\, 2_2 5_1 6_1 7_1 8_1) \end{array}\right.$$

The cones at (5) and (8) are similar to (3), and (6) and (7) are similar to (4). The double quadric is of the σ, τ type. By imposing conditions on the basis points, (3,5) congruences may be determined which have their basis points arranged on ten or twelve planes.

51. In case the planes (1467), (2358) form the double quadric of the net, $K_2(1;\, 3_1 5_1 8_1)$ determines a (3,4) congruence with basis points arranged by fours on eight planes. The complete configuration for this congruence is:

$$\left\{\begin{array}{l} K_2(1;\, 3_1 5_1 8_1) \\ K_2'(1;\, 3_1 5_1 8_1) \end{array}\right. \qquad \left\{\begin{array}{l} K_2(2;\, 4_1 6_1 7_1) \\ K_2'(2;\, 4_1 6_1 7_1) \end{array}\right.$$

$$\left\{\begin{array}{l} K_2(3;\, 1_1 6_1 7_1) \\ K_2'(3;\, 1_1 6_1 7_1) \end{array}\right. \qquad \left\{\begin{array}{l} K_2(4;\, 2_1 5_1 8_1) \\ K_2'(4;\, 2_1 5_1 8_1) \end{array}\right.$$

The cones at (5), (6), (7) and (8) are similar to that at (4). This is the second case in which the σ and τ lines enter in perfect symmetry and in which all the cones are identical in structure.

52. There is but one type of (3,4) congruence, and this may be derived from either I(3,5) or II(3,5). These congruences have each a σ, τ double quadric, and the configurations are:

$$\left\{\begin{array}{l} K_3(1;\, 3_1 4_1 5_1 6_1 8_2) \\ K'_1(1) \end{array}\right. \qquad \left\{\begin{array}{l} K_1(2) \\ K'_3(2;\, 3_1 4_1 5_1 6_1 7_2) \end{array}\right.$$

$$\left\{\begin{array}{l} K_2(3;\, 1_1 6_1 8_1) \\ K'_2(3;\, 2_1 6_1 7_1) \end{array}\right.$$

The cone at (7) is similar to (2); (8) is similar to (1); and (4), (5) and (6) are similar to (3).

By imposing the proper condition, a (3,4) congruence can be found which has its basis points arranged on twelve planes.

53. By imposing the conditions for eight basis points, in the case of a (3,3) congruence, there appears one and only one congruence. This may be obtained from I(3,3), II(3,3) or III(3,3). It has eight basis planes and eight basis points in addition to the twelve singular points and the twelve singular planes. All the configurations of τ lines are plane pencils, and all the configurations of σ lines are quadric cones. The typical form of singular cone is:

$$\left\{\begin{array}{l} K_2(1;\, 3_1 5_1 7_4) \\ K'_1(1) \end{array}\right.$$

It will be noticed that there is no possibility of obtaining a (3,3) with eight basis points on five pairs of planes. This is because the doubling of any one of the five pairs of planes would require at least one of the lines of the congruence to be a fourfold line, which is impossible without a basis line.

54. As a summary of the preceding work, it is seen that a (3,9) congruence may have a double quadric of the $2\sigma, 2\tau$ type or of the σ, τ type, and further may have its points arranged by fours on two, four or six planes.

A (3,8) congruence may have a double quadric of the $2\sigma, 2\tau$ type or the σ, τ type, and have its basis points on i pairs of planes, $i=1, \ldots, 4$.

A (3,7) congruence may have a double quadric of the $2\sigma, 2\tau$ type or the σ, τ type, or may have a pair of planes for a double quadric. It may have its basis points on i pairs of planes, $i=1, \ldots, 5$.

A (3,6) congruence may have a double quadric of the $2\sigma, 2\tau$ type or the σ, τ type, or may have a pair of planes for a double quadric. It may have its basis points arranged on i pairs of planes, $i=2, \ldots, 6$.

A (3,5) congruence may have a σ, τ type of double quadric, or may have a pair of planes for double quadric. It may have its basis points arranged on i pairs of planes, $i=3, \ldots, 6$.

A (3,4) congruence may have a σ, τ type of double quadric, or may have a pair of planes for double quadric. It may have its basis points arranged on i pairs of planes, $i=4, \ldots, 6$.

A (3,3) congruence has a σ, τ type of double quadric and its basis points by fours in six pairs of planes. It contains eight basis planes also.

ITHACA, N. Y., *June*, 1910.